Les Houches lectures on adiabatic potentials Hélène Perrin
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Lecture 1

Spins and fields

Warning: These lecture notes have been written (quickly) as a support of a Les Houches course on adiabatic potentials for rf-dressed atoms. They may still contain some errors. Comments are welcome.

This first lecture is devoted to the interaction of a spin with a magnetic field, first alone, then with an additional radio-frequency field. The effect of these fields is to rotate the spin. When, in addition, the field magnitude or direction depend on position, the question of adiabatic following of the quantization axis determined by the direction of the static magnetic field becomes crucial. In this lecture, we introduce the basic ingredients necessary to understand adiabatic potentials.

The following references may be useful to the reader:

1. on spin-field interaction and on the dressed state approach: a recent book by Cohen-Tannoudji and Guéry-Odelin [1]. This book is also very useful for several other topics covered by the school.

2. on adiabatic potentials: papers by Zobay and Garraway 2004 [2], Lesanovsky et al. 2006 [3]; a review paper by Barry Garraway and myself is in preparation for IOP (ask me...).

3. on spin flips and Landau-Zener transitions: [4], [5].

1 Spin rotation

1.1 Brief reminder on spin operators

A spin operator $\hat{\mathbf{S}}$ is a vector operator describing the spin S of a particle. $S \geq 0$ is an integer for bosonic particles, or a half integer for fermions. The projections of $\hat{\mathbf{S}}$ on any axis $\mathbf{u}$ is $\hat{S}_{\mathbf{u}} = \hat{\mathbf{S}} \cdot \mathbf{u}$, and is an operator in the space of spin vectors.

Remark What we call here spin also apply to angular momentum in general. For the particles having a nucleus spin $\mathbf{I}$, an orbital angular momentum $\mathbf{L}$ and an electronic spin $\mathbf{S}$, the total angular momentum operator relevant to the interaction with weak magnetic field is $\mathbf{F} = \mathbf{J} + \mathbf{I} = \mathbf{L} + \mathbf{S} + \mathbf{I}$. For example, for rubidium 87 atoms in their $5S_{1/2}$ ground state, we have $I = 3/2$, $L = 0$ and $S = 1/2$, such that $J = 1/2$ and $F \in \{|I - J|, ...|I + J|\}$: $F = 1$ or $F = 2$, which are the two hyperfine states of the atomic ground state. For the purpose of this lecture, where spins will interact with static magnetic field or radio-frequency fields, the angular momentum we must consider is a fixed F. In the following, we will use $\hat{\mathbf{S}}$ as the spin notation, which must be understood as $\hat{\mathbf{F}}$ in the case of an alkali atom in its ground state.

Given a quantization axis $\mathbf{e}_z$, we can find a basis where both $\hat{\mathbf{S}}^2$ and $\hat{S}_z$ are diagonal. The spin eigenstates are labelled $|S, m\rangle$ where $m \in \{-S, -S+1, \ldots, S-1, S\}$, with eigenenergies given by

$$\hat{\mathbf{S}}^2 |S, m\rangle = S(S+1)\hbar^2 |S, m\rangle, \tag{1}$$
$$\hat{S}_z |S, m\rangle = m\hbar |S, m\rangle. \tag{2}$$

The other projections of $\hat{\mathbf{S}}$ in a orthogonal basis of axes (x, y, z), however, are not diagonal in this basis. The spin projection operators verify the following commutation relations:

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z, \qquad [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x, \qquad [\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y. \tag{3}$$

We also introduce the rising and lowering operators $\hat{S}_+$ and $\hat{S}_-$, defined as

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y. \tag{4}$$

It is clear from their definition that $[\hat{S}_\pm]^\dagger = \hat{S}_\mp$. Their commutation relations with $\hat{S}_z$ are:

$$[\hat{S}_z, \hat{S}_\pm] = \pm\hbar \hat{S}_\pm. \tag{5}$$

From these relations, we can deduce their effect on $|S, m\rangle$, which is to increase (resp. decrease) m by one unit:

$$\hat{S}_\pm |S, m\rangle = \hbar\sqrt{S(S+1) - m(m \pm 1)}\,|S, m \pm 1\rangle. \tag{6}$$

1.2 Spin rotation operators

From now on, as we will concentrate on operators with do not change the value of S, we will simplify the spin state notation and use $|m\rangle$, where the spin number S is implicit, or $|m\rangle_z$ to emphasize that the quantization axis is chosen along z. Conversely, an eigenstate of $\hat{S}_\mathbf{u}$ will be labeled $|m\rangle_\mathbf{u}$.

The operator which allows to transform $|m\rangle_z$ into $|m\rangle_{z'}$ where z' is a new quantization axis is a rotation operator. The rotation around any axis $\mathbf{u}$ by an angle α is described by the unitary operator

$$\hat{R}_\mathbf{u}(\alpha) = \exp\left[-\frac{i}{\hbar}\alpha \hat{\mathbf{S}} \cdot \mathbf{u}\right]. \tag{7}$$

The inverse rotation, by an angle $-\alpha$, is described by its hermitien conjugate: $[\hat{R}_\mathbf{u}(\alpha)]^\dagger \hat{R}_\mathbf{u}(\alpha) = \mathbb{1}$. Starting from an eigenstate $|m\rangle_z$ of $\hat{S}_z$, the effect of $\hat{R} = \hat{R}_\mathbf{u}(\alpha)$ is to give the corresponding eigenstate $|m\rangle_{z'} = \hat{R}^\dagger |m\rangle_z$ of the rotated operator $\hat{S}_{z'} = \hat{R}^\dagger \hat{S}_z \hat{R}$:

$$|m\rangle_z = \hat{R}|m\rangle_{z'} \Rightarrow \hat{S}_{z'}|m\rangle_{z'} = \hat{R}^\dagger \hat{S}_z \hat{R}|m\rangle_{z'} = \hat{R}^\dagger \hat{S}_z |m\rangle_z = m\hbar \hat{R}^\dagger |m\rangle_z = |m\rangle_{z'}.$$

The rotation by the sum of two angles is simply the product of the two rotations:

$$\hat{R}_\mathbf{u}(\alpha + \beta) = \hat{R}_\mathbf{u}(\alpha)\hat{R}_\mathbf{u}(\beta).$$

However, as the spin projections do not commute, the composition of rotations around different axes do not commute. A useful formula is the decomposition of a rotation around any vector $\mathbf{u}$ in terms of rotations around the basis axes (x, y, z). If the spherical angles describing the direction of the unit vector $\mathbf{u}$ are θ, ϕ such that

$$\mathbf{u} = \sin\theta\,(\cos\phi\mathbf{e}_x + \sin\phi\mathbf{e}_y) + \cos\theta\mathbf{e}_z, \tag{8}$$

we can write

$$\hat{R}_{\mathbf{u}}(\alpha) = \hat{R}_z(\phi)\hat{R}_y(\theta)\hat{R}_z(\alpha)[\hat{R}_y(\theta)]^\dagger[\hat{R}_z(\phi)]^\dagger, \tag{9}$$

or

$$\hat{R}_{\mathbf{u}}(\alpha) = \hat{R}_z(\phi)\hat{R}_y(\theta)\hat{R}_z(\alpha)\hat{R}_y(-\theta)\hat{R}_z(-\phi), \tag{10}$$

where $\hat{R}_i$ stands for $\hat{R}_{\mathbf{e}_i}$. Starting from the right hand side, the two first rotations put $\mathbf{u}$ on top of z, the central operator makes the rotation by α around z, and the two last operators bring back $\mathbf{u}$ to its original position.

Important remark If we look at the rotation by 2π around any axis, we find that its effect on a state $|m\rangle_{\mathbf{u}}$ is

$$\hat{R}_{\mathbf{u}}(2\pi)|m\rangle_{\mathbf{u}} = e^{-i\frac{2\pi}{\hbar}\hat{S}_{\mathbf{u}}}|m\rangle_{\mathbf{u}} = e^{-i2\pi m}|m\rangle_{\mathbf{u}} = (-1)^{2m}|m\rangle_{\mathbf{u}}.$$

For integer spins, $2m$ is even and the final state is the same as the initial state: the 2π rotation is identity. For an odd spin, however, the final state is the opposite of the initial state. We need to rotate by 4π to recover identity. This is linked to the spin statistics theorem.

1.3 Rotation of usual spin operators

In the lecture, we will need to transform hamiltonians $\hat{H}$ through rotations, calculating operators such as $\hat{R}^\dagger\hat{H}\hat{R}$. In order to become more familiar with this transformation, we give here its effect in simple cases. Let us first consider rotations by α around the quantization axis z, such that $\hat{R} = \hat{R}_z(\alpha)$.

$$\left[\hat{R}_z(\alpha)\right]^\dagger \hat{S}_z\hat{R}_z(\alpha) = \hat{S}_z, \tag{11}$$

$$\left[\hat{R}_z(\alpha)\right]^\dagger \hat{S}_\pm\hat{R}_z(\alpha) = e^{\pm i\alpha}\hat{S}_\pm, \tag{12}$$

$$\left[\hat{R}_z(\alpha)\right]^\dagger \hat{S}_x\hat{R}_z(\alpha) = \cos\alpha\,\hat{S}_x - \sin\alpha\,\hat{S}_y, \tag{13}$$

$$\left[\hat{R}_z(\alpha)\right]^\dagger \hat{S}_y\hat{R}_z(\alpha) = \sin\alpha\,\hat{S}_x + \cos\alpha\,\hat{S}_y. \tag{14}$$

By circular permutation, it is clear that we can also write, for rotations around x and y:

$$\left[\hat{R}_x(\alpha)\right]^\dagger \hat{S}_y\hat{R}_x(\alpha) = \cos\alpha\,\hat{S}_y - \sin\alpha\,\hat{S}_z, \tag{15}$$

$$\left[\hat{R}_x(\alpha)\right]^\dagger \hat{S}_z\hat{R}_x(\alpha) = \sin\alpha\,\hat{S}_y + \cos\alpha\,\hat{S}_z, \quad \text{and} \tag{16}$$

$$\left[\hat{R}_y(\alpha)\right]^\dagger \hat{S}_z\hat{R}_y(\alpha) = \cos\alpha\,\hat{S}_z - \sin\alpha\,\hat{S}_x, \tag{17}$$

$$\left[\hat{R}_y(\alpha)\right]^\dagger \hat{S}_x\hat{R}_y(\alpha) = \sin\alpha\,\hat{S}_z + \cos\alpha\,\hat{S}_x. \tag{18}$$

The effect of a rotation of $\hat{S}_\pm$ around x or y is a bit more complicated, but directly deduced from these equations and the definition of $\hat{S}_\pm$:

$$\left[\hat{R}_x(\alpha)\right]^\dagger \hat{S}_+ \hat{R}_x(\alpha) = \cos^2\frac{\alpha}{2}\,\hat{S}_+ + \sin^2\frac{\alpha}{2}\,\hat{S}_- - i\sin\alpha\,\hat{S}_z, \tag{19}$$

$$\left[\hat{R}_x(\alpha)\right]^\dagger \hat{S}_- \hat{R}_x(\alpha) = \sin^2\frac{\alpha}{2}\,\hat{S}_+ + \cos^2\frac{\alpha}{2}\,\hat{S}_- + i\sin\alpha\,\hat{S}_z, \quad \text{and} \tag{20}$$

$$\left[\hat{R}_y(\alpha)\right]^\dagger \hat{S}_+ \hat{R}_y(\alpha) = \cos^2\frac{\alpha}{2}\,\hat{S}_+ - \sin^2\frac{\alpha}{2}\,\hat{S}_- + \sin\alpha\,\hat{S}_z, \tag{21}$$

$$\left[\hat{R}_y(\alpha)\right]^\dagger \hat{S}_- \hat{R}_y(\alpha) = -\sin^2\frac{\alpha}{2}\,\hat{S}_+ + \cos^2\frac{\alpha}{2}\,\hat{S}_- + \sin\alpha\,\hat{S}_z. \tag{22}$$

1.4 Two exercises

Exercise 1 Calculate the transformed operator $\hat{S}_\mathbf{u} = \hat{\mathbf{S}} \cdot \mathbf{u}$ under the rotation $\hat{R}_z(\alpha)$.

Answer:

$$\begin{aligned}
&[\hat{R}_z(\alpha)]^\dagger \hat{S}_\mathbf{u} \hat{R}_z(\alpha) \\
&= [\hat{R}_z(\alpha)]^\dagger \left(\sin\theta\cos\phi\,\hat{S}_x + \sin\theta\sin\phi\,\hat{S}_y + \cos\theta\,\hat{S}_z\right) \hat{R}_z(\alpha) \\
&= \sin\theta\cos\phi\left(\cos\alpha\,\hat{S}_x - \sin\alpha\,\hat{S}_y\right) + \sin\theta\sin\phi\left(\cos\alpha\,\hat{S}_y + \sin\alpha\,\hat{S}_x\right) + \cos\theta\,\hat{S}_z \\
&= \sin\theta\left(\cos\phi\cos\alpha + \sin\phi\sin\alpha\right)\hat{S}_x + \sin\theta\left(\sin\phi\cos\alpha - \cos\phi\sin\alpha\right)\hat{S}_y + \cos\theta\,\hat{S}_z \\
&= \sin\theta\cos(\phi-\alpha)\,\hat{S}_x + \sin\theta\sin(\phi-\alpha)\,\hat{S}_y + \cos\theta\,\hat{S}_z.
\end{aligned}$$

The result is quite intuitive: the transformed projection is the projection on a unit vector whose azimuthal angle has changed from ϕ to $\phi - \alpha$.

Exercise 2 Calculate the transformed operator $\hat{S}_{z'}$ of $\hat{S}_z$ through the rotation $\hat{R}_\mathbf{u}(\alpha)$.

Answer: You could replace $\hat{R}_\mathbf{u}(\alpha)$ by its expression in terms of elementary rotations, but this would be a nightmare... There is a much clever trick: first write $\mathbf{e}_z$ in the $(\mathbf{u}, \mathbf{u}_\theta, \mathbf{u}_\phi)$ orthonormal basis, where $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$ are defined by

$$\mathbf{u}_\theta = \cos\theta(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) - \sin\theta\,\mathbf{e}_z \tag{23}$$

$$\mathbf{u}_\phi = -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y \tag{24}$$

Within this new basis, we can write $\mathbf{e}_z = \cos\theta\,\mathbf{u} - \sin\theta\,\mathbf{u}_\theta$, such that

$$\hat{S}_z = \cos\theta\,\hat{S}_\mathbf{u} - \sin\theta\,\hat{S}_{\mathbf{u}_\theta}.$$

Then use the rotation formulae, with the correspondence $\mathbf{u} \leftrightarrow \mathbf{e}_z$, $\mathbf{u}_\theta \leftrightarrow \mathbf{e}_x$, $\mathbf{u}_\phi \leftrightarrow \mathbf{e}_y$:

$$\begin{aligned}
[\hat{R}_\mathbf{u}(\alpha)]^\dagger \hat{S}_z \hat{R}_\mathbf{u}(\alpha) &= [\hat{R}_\mathbf{u}(\alpha)]^\dagger \left(\cos\theta\,\hat{S}_\mathbf{u} - \sin\theta\,\hat{S}_{\mathbf{u}_\theta}\right) \hat{R}_\mathbf{u}(\alpha) \\
&= \cos\theta\,\hat{S}_\mathbf{u} - \sin\theta[\hat{R}_\mathbf{u}(\alpha)]^\dagger \hat{S}_{\mathbf{u}_\theta} \hat{R}_\mathbf{u}(\alpha) \\
&= \cos\theta\,\hat{S}_\mathbf{u} - \sin\theta\left(\cos\alpha\,\hat{S}_{\mathbf{u}_\theta} - \sin\alpha\,\hat{S}_{\mathbf{u}_\phi}\right).
\end{aligned}$$

That's it! With these two exercises, you're ready to perform any rotation in the spin space.

1.5 Time-dependent rotations

We will need to deal with time-dependent rotation angles, and with the derivatives or R operators. Let us look into this here.

Let us consider first the simple case where the rotation axis, $\mathbf{u}$, is fixed. The time derivative of the rotation operator is

$$i\hbar\partial_t \hat{R}_{\mathbf{u}}(\alpha(t)) = i\hbar\partial_t e^{\frac{i}{\hbar}\alpha(t)\hat{S}_{\mathbf{u}}} = \dot{\alpha}\hat{S}_{\mathbf{u}}\, e^{\frac{i}{\hbar}\alpha(t)\hat{S}_{\mathbf{u}}} = \dot{\alpha}\hat{S}_{\mathbf{u}}\hat{R}_{\mathbf{u}}(\alpha) = \dot{\alpha}\hat{R}_{\mathbf{u}}(\alpha)\hat{S}_{\mathbf{u}}. \tag{25}$$

This expression is simple, because the rotation operator $\hat{R}_{\mathbf{u}}(\alpha)$ commutes with the spin projection along the rotation axis $\hat{S}_{\mathbf{u}}$.

The situation is different if the rotation axis itself is time-dependent. This is a relevant case for adiabatic potentials, but also for magnetic traps, where the natural quantization axis, aligned with the static magnetic field, depends on position, and hence on time when the atom moves in the trap. The vector $\mathbf{u}$ evolves with time:

$$\dot{\mathbf{u}} = \dot{\theta}\,\mathbf{u}_\theta + \dot{\phi}\sin\theta\,\mathbf{u}_\phi.$$

By writing carefully $i\hbar\partial_t \hat{R}_{\mathbf{u}}(\alpha)$ as

$$i\hbar\partial_t \hat{R}_{\mathbf{u}}(\alpha) = i\hbar \lim_{\tau\to 0}\frac{1}{\tau}\left(e^{\frac{i}{\hbar}\alpha(t+\tau)\hat{\mathbf{S}}\cdot\mathbf{u}(\mathbf{t}+\tau)} - e^{\frac{i}{\hbar}\alpha(t)\hat{\mathbf{S}}\cdot\mathbf{u}(\mathbf{t})}\right),$$

we see that terms involving spin projections $\hat{S}_{\mathbf{u}_\theta}$ and $\hat{S}_{\mathbf{u}_\phi}$ appear in the argument of the exponential, which do not commute with $\hat{R}_{\mathbf{u}}(\alpha)$ anymore.

We can find the time derivative of $\hat{R}_{\mathbf{u}}(\alpha)$ by using the decomposition (10) in rotations around fixed axes. The variations of α give the same expression as above, whereas the variations of $\mathbf{u}$ introduce commutators:

$$\begin{aligned}
i\hbar\partial_t \hat{R}_{\mathbf{u}}(\alpha) &= \dot{\alpha}\hat{R}_{\mathbf{u}}(\alpha)\hat{S}_{\mathbf{u}} + \dot{\phi}\left[S_z, \hat{R}_{\mathbf{u}}(\alpha)\right] \\
&+ \dot{\theta}\left(R_z(\phi)\left[\hat{S}_y, \hat{R}_y(\theta)\hat{R}_z(\alpha)\hat{R}_y(-\theta)\right]R_z(-\phi)\right) \\
i\hbar\partial_t \hat{R}_{\mathbf{u}}(\alpha) &= \dot{\alpha}\hat{R}_{\mathbf{u}}(\alpha)\hat{S}_{\mathbf{u}} + \dot{\phi}\left[S_z, \hat{R}_{\mathbf{u}}(\alpha)\right] + \dot{\theta}\left[S_y, \hat{R}_{\mathbf{u}}(\alpha)\right] \\
&- \dot{\theta}\left[\hat{S}_y, R_z(\phi)\right]\hat{R}_y(\theta)\hat{R}_z(\alpha)\hat{R}_y(-\theta)R_z(-\phi) \\
&- \dot{\theta}R_z(\phi)\hat{R}_y(\theta)\hat{R}_z(\alpha)\hat{R}_y(-\theta)\left[\hat{S}_y, R_z(-\phi)\right] \\
i\hbar\partial_t \hat{R}_{\mathbf{u}}(\alpha) &= \dot{\alpha}\hat{R}_{\mathbf{u}}(\alpha)\hat{S}_{\mathbf{u}} + \dot{\phi}\left[S_z, \hat{R}_{\mathbf{u}}(\alpha)\right] + \dot{\theta}\left[S_y, \hat{R}_{\mathbf{u}}(\alpha)\right] \\
&+ \dot{\theta}(\cos\phi - 1)\left[\hat{S}_y, \hat{R}_{\mathbf{u}}(\alpha)\right] - \dot{\theta}\sin\phi\left[\hat{S}_x, \hat{R}_{\mathbf{u}}(\alpha)\right] \\
i\hbar\partial_t \hat{R}_{\mathbf{u}}(\alpha) &= \dot{\alpha}\hat{R}_{\mathbf{u}}(\alpha)\hat{S}_{\mathbf{u}} + \dot{\phi}\left[S_z, \hat{R}_{\mathbf{u}}(\alpha)\right] \\
&+ \dot{\theta}\left[\cos\phi\hat{S}_y - \sin\phi\hat{S}_x, \hat{R}_{\mathbf{u}}(\alpha)\right].
\end{aligned}$$

Using the $(\mathbf{u}, \mathbf{u}_\theta, \mathbf{u}_\phi)$ basis at time t, we recognize $\hat{S}_{\mathbf{u}_\phi}$ in the last commutator, and we decompose $\hat{S}_z$ in this basis, remembering that $\hat{S}_{\mathbf{u}}$ commutes with $\hat{R}_{\mathbf{u}}(\alpha)$:

$$i\hbar\partial_t \hat{R}_{\mathbf{u}}(\alpha) = \dot{\alpha}\hat{R}_{\mathbf{u}}(\alpha)\hat{S}_{\mathbf{u}} + \dot{\theta}\left[\hat{S}_{\mathbf{u}_\phi}, \hat{R}_{\mathbf{u}}(\alpha)\right] - \dot{\phi}\sin\theta\left[S_{\mathbf{u}_\theta}, \hat{R}_{\mathbf{u}}(\alpha)\right]. \tag{26}$$

We will use this expression, which we recast under the form:

$$i\hbar\hat{R}^\dagger_{\mathbf{u}}(\alpha)\partial_t\hat{R}_{\mathbf{u}}(\alpha) = \dot{\alpha}\hat{S}_{\mathbf{u}} + (1-\cos\alpha)\left[-\dot{\theta}\hat{S}_{\mathbf{u}_\phi} + \dot{\phi}\sin\theta\hat{S}_{\mathbf{u}_\theta}\right] + \sin\alpha\left[\dot{\phi}\sin\theta\hat{S}_{\mathbf{u}_\phi} + \dot{\theta}\hat{S}_{\mathbf{u}_\theta}\right]. \tag{27}$$

The important message is that, if the direction $\mathbf{u}$ around which the rotation is performed varies with time, the time derivative of the rotation operator now involves also spin projections along directions orthogonal to $\mathbf{u}$.

2 Spin in a static magnetic field

2.1 Magnetic interaction

The interaction between a spin[1] $\hat{\mathbf{S}}$ and a static magnetic field $\mathbf{B}_0$ writes

$$\hat{H} = -\gamma\hat{\mathbf{S}}\cdot\mathbf{B}_0, \tag{28}$$

where $\gamma = -\frac{g_S\mu_B}{\hbar}$ is the gyromagnetic ratio, g_S is the Landé factor and μ_B is the Bohr magneton.

The eigenstates of $\hat{H}$ are the states $|m\rangle_{\mathbf{u}}$, eigenstates of $\hat{\mathbf{S}}\cdot\mathbf{u}$, where $\mathbf{B}_0 = B_0\mathbf{u}$. If the z axis is chosen along $\mathbf{B}_0$, these states are $|m\rangle_z$. The corresponding eigenenergies are

$$E_m = mg_S\mu_B B_0. \tag{29}$$

2.2 Position dependent magnetic fields. Magnetic traps

If the magnetic field amplitude and direction depends on position, we must consider the total hamiltonian, including the external degrees of freedom of an atom of mass M:

$$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2M} + \frac{g_s\mu_B}{\hbar}\hat{\mathbf{S}}\cdot\mathbf{B}_0(\hat{\mathbf{R}}).$$

We could find the spin eigenstate at each fixed position $\mathbf{r} = \langle\hat{\mathbf{R}}\rangle$. If the magnetic field direction is space dependent, the quantization axis now changes as the atom moves. In a first approach, we can then just identify the internal energy in state $|m\rangle_{\mathbf{u}(\mathbf{r})}$ with $mg_S\mu_B B_0(\mathbf{r})$. The Larmor frequency $\omega_0(\mathbf{r}) = g_S\mu_B B_0(\mathbf{r})/\hbar$ depends on position. Such a position dependent magnetic field is used to create a magnetic potential to trap atoms in magnetic traps. The Zeeman states such that $mg_S > 0$, called the low-field seekers, are trapped to a minimum of the modulus of the magnetic field.

However, as the position operator $\hat{\mathbf{R}}$ and the momentum operator $\hat{\mathbf{P}}$ do not commute, the spin eigenstate at a given position is coupled to other spin states. This effect is known as Majorana spin flips, from the italian physicist Ettore Majorana [6].

2.3 Majorana spin flips

In order to understand more clearly this effect, let us introduce explicitly the transformation which diagonalizes the magnetic interaction at each point. This transformation

[1]Again, S can be a total spin F, in the limit of weak magnetic fields.

brings the vector $\mathbf{u}$ directing the magnetic field onto z, it is a rotation. There are several possible choices for this rotation, but we can choose the rotation by an angle π around the direction $\mathbf{u}'$ with bisects $(\mathbf{e}_z, \mathbf{u})$.

$$\mathbf{u}' = \sin\frac{\theta}{2}\cos\phi\,\mathbf{e}_x + \sin\frac{\theta}{2}\sin\phi\,\mathbf{e}_y + \cos\frac{\theta}{2}\,\mathbf{e}_z.$$

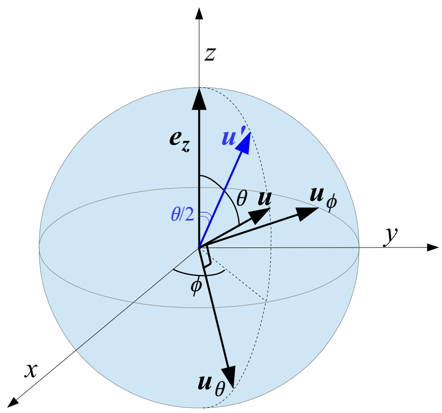

Figure 1: Orientation of the direction of the magnetic field $\mathbf{u}$, and vector $\mathbf{u}'$ around which a π rotation transforms $\mathbf{e}_z$ into $\mathbf{u}$ and vice versa.

Note that $\mathbf{u'}_\phi = \mathbf{u}_\phi$ Eq.(24), and

$$\mathbf{u}'_\theta = \cos\frac{\theta}{2}\cos\phi\,\mathbf{e}_x + \cos\frac{\theta}{2}\sin\phi\,\mathbf{e}_y - \sin\frac{\theta}{2}\,\mathbf{e}_z.$$

We have then

$$\hat{S}_z = \hat{R}^\dagger_{\mathbf{u}'}(\pi)\hat{S}_{\mathbf{u}}\hat{R}_{\mathbf{u}'}(\pi).$$

The loss rate due to Majorana transition in a Ioffe Pritchard magnetic trap was calculated by Sukumar and Brinks [4, 7]. Their approach, which is very general and also holds for unitary transformations other than rotations, is the following: we know the unitary transformation $U(\mathbf{r}) = \hat{R}_{\mathbf{u}'}(\pi)$ which at each point $\mathbf{r}$ transforms the magnetic interaction into a pure $\hat{S}_z$ operator:

$$U^\dagger(\hat{\mathbf{R}})\frac{g_s\mu_B}{\hbar}\hat{\mathbf{S}}\cdot\mathbf{B}_0(\hat{\mathbf{R}})U(\hat{\mathbf{R}}) = \frac{g_s\mu_B B_0(\hat{\mathbf{R}})}{\hbar}\hat{S}_z.$$

The transform of the full hamiltonian $\hat{H} = \frac{\hat{\mathbf{P}}^2}{2M} + \frac{g_s\mu_B}{\hbar}\hat{\mathbf{S}}\cdot\mathbf{B}_0(\hat{\mathbf{R}})$ is $U^\dagger(\hat{\mathbf{R}})\hat{H}U(\hat{\mathbf{R}})$ and also contains the transform of the kinetic energy $T = \frac{\hat{\mathbf{P}}^2}{2M}$:

$$T' = U^\dagger(\hat{\mathbf{R}})TU(\hat{\mathbf{R}}) = T + \left[U^\dagger(\hat{\mathbf{R}})TU(\hat{\mathbf{R}}) - T\right] = T + \Delta T.$$

The final transform is thus

$$U^\dagger(\hat{\mathbf{R}})\hat{H}U(\hat{\mathbf{R}}) = T + \frac{g_s\mu_B B_0(\hat{\mathbf{R}})}{\hbar}\hat{S}_z + \Delta T.$$

If we neglect the term ΔT, we assume that the spin follows adiabatically the local direction of the magnetic field, and is in the state $|m\rangle_{\mathbf{u}(\mathbf{r})}$. This is known as the adiabatic approximation, and is valid if $\langle \Delta T\rangle \ll g_S\mu_B B_0(\mathbf{r})$.

Let us examine the transform of $\hat{\mathbf{P}}$:

$$\hat{\mathbf{P}}' = U^\dagger(\hat{\mathbf{R}})\hat{\mathbf{P}}U(\hat{\mathbf{R}}) = \hat{\mathbf{P}} - i\hbar U^\dagger(\hat{\mathbf{R}})\left[\boldsymbol{\nabla} U(\hat{\mathbf{R}})\right] = \hat{\mathbf{P}} + \hat{\mathbf{A}}.$$

From the same approach (27) we used for time derivative, we can calculate the gradient of the rotation operator U, with angles $\alpha = \pi$, $\theta/2$ and ϕ. We find

$$\hat{\mathbf{A}} = -i\hbar U^\dagger(\hat{\mathbf{R}})\left[\boldsymbol{\nabla} U(\hat{\mathbf{R}})\right] = \boldsymbol{\nabla}\theta \hat{S}_{\mathbf{u}'_\phi} - 2\boldsymbol{\nabla}\phi \sin\frac{\theta}{2}\,\hat{S}_{\mathbf{u}'_\theta}.$$

The transitions between spin states induced by $\hat{\mathbf{A}}$ are indeed due to the way the field rotates when the atoms move. The transformed kinetic energy is

$$T' = \frac{\hat{\mathbf{P}}'^2}{2M} = \frac{1}{2M}\left(\hat{\mathbf{P}} + \hat{\mathbf{A}}\right)^2 = T + \frac{1}{2M}\left(\hat{\mathbf{P}}\cdot\hat{\mathbf{A}} + \hat{\mathbf{A}}\cdot\hat{\mathbf{P}}\right) + \frac{\hat{\mathbf{A}}^2}{2M}.$$

Then

$$\Delta T = \frac{1}{2M}\left(\hat{\mathbf{P}}\cdot\hat{\mathbf{A}} + \hat{\mathbf{A}}\cdot\hat{\mathbf{P}}\right) + \frac{\hat{\mathbf{A}}^2}{2M} = \hat{\mathbf{A}}\cdot\frac{\hat{\mathbf{P}}}{M} - i\hbar(\boldsymbol{\nabla}\cdot\hat{\mathbf{A}}) + \frac{\hat{\mathbf{A}}^2}{2M}.$$

ΔT is a small correction typically if $\hat{\mathbf{A}}\cdot\hat{\mathbf{P}}/M$ is a small correction to the energy, of order $\hbar\omega_0(\mathbf{r})$. Classically, this gives

$$|\mathbf{v}\cdot\boldsymbol{\nabla}\theta| = \left|\frac{d\theta}{dt}\right| \ll \omega_0(\mathbf{r}). \tag{30}$$

The magnetic field direction must change slowly as the atom moves.

Brinks and Sukumar evaluated the loss rate from this coupling outside $|m\rangle_z$ to an untrapped plane wave state with a Fermi golden rule, in the case of a Ioffe-Pritchard trap.[2] If the initial and the final external states are labelled respectively $|\varphi_i, m_z\rangle$ and $|\varphi_f, 0\rangle$, the loss rate to a plane wave state is

$$\Gamma_{\mathrm{Maj}} = \frac{2\pi}{\hbar}\,|\langle\varphi_f, 0|\Delta T|\varphi_i, m_z\rangle|^2\,\rho_f\left(m_z\hbar\omega_0\right),$$

where ρ_f is the density of states in the final plane wave state.

The coupling term matrix element can be deduced from the knowledge of $\hat{\mathbf{A}}$, which is calculated from the spatial dependence of the magnetic field. A Ioffe-Pritchard trap is characterized by a magnetic gradient b', corresponding to a Zeeman shift gradient $\alpha = |g_S|\mu_B b'/\hbar$, and a Larmor frequency at the trap bottom $\omega_0 = |g_S|\mu_B B_0/\hbar$. The magnetic field close to the trap bottom reads

$$\mathbf{B}_0(\mathbf{r}) = B_0\,\mathbf{e}_z + b'(x\,\mathbf{e}_x - y\,\mathbf{e}_y).$$

We have neglected the slow longitudinal dependence on z, which will not be the major source of Majorana transitions. I let the calculation of the matrix element as a (lengthy) exercise.

[2] We chose an even S, so that the final state is $m_z = 0$ and the external state is a plane wave.

For a $S = 1$ spin, where a single spin flip results in a loss, the result is

$$\Gamma_{\mathrm{Maj}} = \pi\omega_{\mathrm{osc}}\, e^{-\frac{2\omega_0}{\omega_{\mathrm{osc}}}} \tag{31}$$

where the oscillation frequency in the trap is $\omega_{\mathrm{osc}} = \alpha\sqrt{\hbar/(M\omega_0)}$. The coefficient in the exponential scales as

$$\frac{2\omega_0}{\omega_{\mathrm{osc}}} \propto \frac{\omega_0^{3/2}}{\alpha}. \tag{32}$$

We will see that this also gives the loss rate from an adiabatic potential due to Landau-Zener losses.

3 Spin in a radio-frequency field

In this section, we limit ourselves to a homogeneous, static magnetic field $\mathbf{B} = B_0\mathbf{e}_z$.

3.1 Effect of an rf field

We now introduce a magnetic field oscillating at a radio-frequency ω on the order of the Larmor frequency $\omega_0 = |\gamma|B_0$. In this section, we describe the rf by a classical magnetic field. In a first, quick, approach, let us consider a homogeneous, linearly polarized rf field along the direction $\mathbf{e}_x$ orthogonal to the static field, also called σ-polarization:

$$\mathbf{B}_1(t) = B_1\cos(\omega t)\,\mathbf{e}_x. \tag{33}$$

The origin of time is chosen arbitrarily to cancel the phase in the cosine. The coupling of this oscillatory field with the spin is again magnetic coupling:

$$\hat{V}_{\mathrm{rf}} = -\gamma\hat{\mathbf{S}}\cdot[B_1\cos(\omega t)\,\mathbf{e}_x] = \frac{g_S\mu_B B_1}{\hbar}\cos\omega t\,\hat{S}_x. \tag{34}$$

Let us introduce the Rabi frequency

$$\Omega = \frac{|g_S|\mu_B B_1}{2\hbar}. \tag{35}$$

The total spin hamiltonian, including the effect of both fields, static and oscillating, reads:

$$\hat{H} = \varepsilon\omega_0\hat{S}_z + \varepsilon 2\Omega\cos\omega t\,\hat{S}_x. \tag{36}$$

Here, $\varepsilon = \pm 1$ is the sign of the Landé factor g_S.

Using the $\hat{S}_\pm$ operators, this can be written as

$$\hat{H} = \varepsilon\omega_0\hat{S}_z + \varepsilon\frac{\Omega}{2}\left[e^{-i\omega t}\,\hat{S}_+ + e^{i\omega t}\,\hat{S}_- + e^{-i\omega t}\,\hat{S}_- + e^{i\omega t}\hat{S}_+\right].$$

The first term is responsible for a spin precession around the z axis at frequency ω_0, a in direction determined by ε. The other terms couple different $|m\rangle_z$ states and induce transitions. These transitions will be resonant for $\omega = \omega_0$. To emphasize this point, let us write the hamiltonian in the basis rotating at $\varepsilon\omega$ around z: we introduce the state $|\psi'\rangle$ such that, if $|\psi\rangle$ satisfies the time-dependent Schrödinger equation with $\hat{H}$, $|\psi\rangle = \hat{R}_z(\varepsilon\omega t)|\psi'\rangle$. We write the Schrödinger equation for $|\psi\rangle$:

$$i\hbar\partial_t|\psi\rangle = i\hbar\left[\partial_t\hat{R}\right]|\psi'\rangle + \hat{R}\left[i\hbar\partial_t|\psi'\rangle\right] = \hat{H}|\psi\rangle = \hat{H}\hat{R}|\psi'\rangle.$$

$$i\hbar\partial_t|\psi'\rangle = -i\hbar\hat{R}^\dagger\left[\partial_t\hat{R}\right]|\psi'\rangle + \hat{R}^\dagger\hat{H}\hat{R}|\psi'\rangle.$$

$|\psi'\rangle$ is thus governed by an effective hamiltonian

$$\hat{H}_{\text{eff}} = -i\hbar\hat{R}^\dagger\left[\partial_t\hat{R}\right] + \hat{R}^\dagger\hat{H}\hat{R}. \tag{37}$$

The value of the first term is given by Eq.(27): $-\varepsilon\omega\hat{S}_z$. The second is simply the rotated hamiltonian. Introducing the detuning $\delta = \omega - \omega_0$, we get

$$\hat{H}_{\text{eff}} = -\varepsilon\delta\hat{S}_z + \varepsilon\frac{\Omega}{2}\left[e^{i(\varepsilon-1)\omega t}\,\hat{S}_+ + e^{-i(\varepsilon-1)\omega t}\,\hat{S}_- + e^{-(\varepsilon+1)i\omega t}\,\hat{S}_- + e^{i(\varepsilon+1)\omega t}\hat{S}_+\right]. \tag{38}$$

Depending on the sign of ε, either the two first terms ($\varepsilon = 1$) or the two last terms ($\varepsilon = -1$) in the bracket become static. On the other hand, the two other terms evolve at high frequency $\pm 2\omega$.

3.2 Rotating wave approximation

We now proceed to an important approximation, called the rotating wave approximation or RWA, which consists in neglecting the effect of the fast oscillatory terms at ω in front of the static terms, which will lead to an evolution with a time scale of order $\sqrt{\delta^2+\Omega^2}$. This is valid if $\sqrt{\delta^2+\Omega^2} \ll \omega$. We point out however that for radiofrequency fields, where $\omega/2\pi$ in the experiments is typically between 100 kHz and 10 MHz, it is not always true that $\Omega \ll \omega$. We will discuss the beyond RWA case in Lecture 3. We will see in section 4 that the neglected terms describe non resonant processes between different manifolds of the dressed atom.

If we apply RWA, the effective hamiltonian simplifies into

$$\hat{H}_{\text{eff}} = -\varepsilon\delta\hat{S}_z + \varepsilon\frac{\Omega}{2}\left[\hat{S}_+ + \hat{S}_-\right] = -\varepsilon\delta\hat{S}_z + \varepsilon\Omega\hat{S}_x = \sqrt{\delta^2+\Omega^2}\,\hat{\mathbf{S}}\cdot\mathbf{u}. \tag{39}$$

The hamiltonian (39) corresponds to the interaction of a spin with a static effective magnetic field

$$\mathbf{B}_{\text{eff}} = \frac{\hbar\sqrt{\delta^2+\Omega^2}}{g_S\mu_B}\mathbf{u}, \tag{40}$$

where the new quantization axis for the rotating spin is

$$\mathbf{u} = \cos\theta\mathbf{e}_z + \sin\theta\mathbf{e}_x \qquad \cos\theta = \frac{-\varepsilon\delta}{\sqrt{\delta^2+\Omega^2}} \qquad \sin\theta = \frac{\varepsilon\Omega}{\sqrt{\delta^2+\Omega^2}}. \tag{41}$$

The eigenstates of $\hat{H}_{\text{eff}}$ are $|\psi'\rangle = \hat{R}_y(\theta)|m\rangle_z$, with eigenenergies $m\hbar\sqrt{\delta^2+\Omega^2}$.

Frequency sweep and spin inversion: When $|\delta|$ is very large as compared to Ω, the angle θ is 0 or π, depending on the sign of δ, which means that $\mathbf{u}$ is simply $\pm\mathbf{e}_z$. The eigenstates in the presence of the rf field are $|m\rangle_z$, like without rf. However, the correspondence between the $|m\rangle_{\mathbf{u}}$ and the $|m\rangle_z$ states is inverted as the sign of δ is reversed.

This can be used to flip a spin adiabatically with a frequency sweep. Let us consider for example the case $\varepsilon > 0$, and suppose that an atom is initially prepared in the eigenstate $|m\rangle_z$ of the hamiltonian in the static field only. We then ramp up a rf field for $\Omega = 0$ to $\Omega = \Omega_1$ in a sufficiently long time, at a frequency ω such that $\delta < 0$ and $|\delta| \gg \Omega_1$. The angle θ corresponding to the eigenstate in the presence of a rf field is initially 0, and reaches $\theta \simeq \Omega_0/|\delta|$ at the end of the process. The spin in thus in the state $|m\rangle_{\mathbf{u}}$ where $\mathbf{u}$ is almost $\mathbf{e}_z$. The next step is to sweep ω from $\delta < 0$ to $\delta > 0$ and $\delta \gg \Omega_1$. In this process, θ increases from almost 0 to nearly π. Switching off Ω slowly gives $\theta = \pi$, and the final state is $|-m\rangle_z$. The same process also works to invert the spin in the case where $\varepsilon < 0$, except that in the case, the spin follows the state $|-m\rangle_{\mathbf{u}}$.

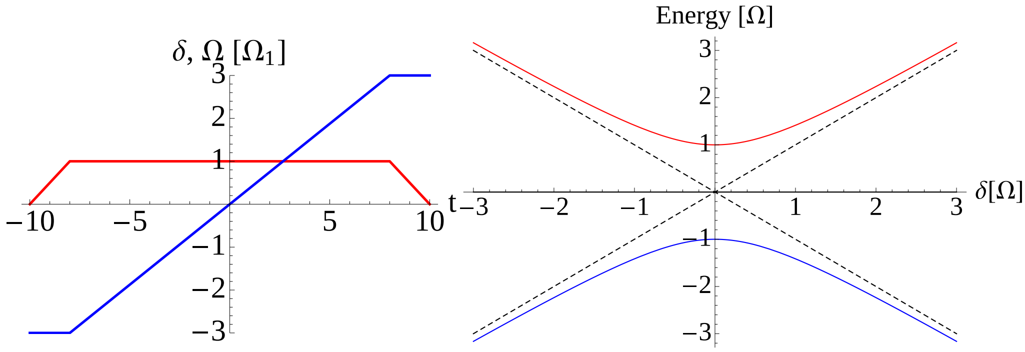

Figure 2: Spin inversion with a sweep of the rf frequency. Left: Evolution of the rf coupling Ω (red line) and the detuning δ (blue line) with time. Right: Corresponding evolution of the eigenenergies in the case of a $S = 1$ spin at the central period where the detuning is varied (upper curve, lower curve and zero line), compared to the uncoupled states $|m\rangle_z$, straight dashed lines. Following a coupled state allows to flip the spin from $+m$, left, to $-m$, right, or vice versa.

This spin flip procedure is efficient if the spin follows adiabatically the eigenstate $|m\rangle_{\mathbf{u}} = \hat{R}_y(\theta)|m\rangle_z$ at any time. This is the case if its variations with time, which provoke a coupling term to the other spin states, are small as compared to the frequency splitting between eigenstates. This gives the condition:

$$|\dot{\theta}| \ll \sqrt{\delta^2 + \Omega^2}, \quad \text{or} \quad \left|\dot{\delta}\Omega - \delta\dot{\Omega}\right| \ll \left(\delta^2 + \Omega^2\right)^{3/2}. \tag{42}$$

This is the *adiabaticity condition*. It is reminiscent of Eq.(30), which expressed the same condition for the spin following of a static field. For the procedure described above, we just have to ensure $|\dot{\Omega}| \ll \delta^2$ for the on and off switching of the rf power, and $|\dot{\delta}| \ll \Omega_1^2$ for the frequency sweep, to fulfill the condition (42).

3.3 Generalization to any polarization

The rf field can be written very generally as

$$\mathbf{B}_1 = B_x \cos(\omega t + \phi_x)\,\mathbf{e}_x + B_y \cos(\omega t + \phi_y)\,\mathbf{e}_y + B_z \cos(\omega t + \phi_z)\,\mathbf{e}_z \tag{43}$$

z being the direction of the static magnetic field. In principle, the amplitudes B_i and phases ϕ_i could depend on position. To start with, we consider a homogeneous rf field.

We now use a complex notation for the field, such that

$$\mathbf{B}_1 = \frac{B_x}{2} e^{-i\phi_x} e^{-i\varepsilon\omega t}\,\mathbf{e}_x + \frac{B_y}{2} e^{-i\phi_y} e^{-i\varepsilon\omega t}\,\mathbf{e}_y + \frac{B_z}{2} e^{-i\phi_z} e^{-i\varepsilon\omega t}\,\mathbf{e}_z + c.c. \tag{44}$$

The z component of the rf field, aligned along the quantization axis, doesn't couple the $|m\rangle_z$ state. We will discard this term[3] from now on.

We introduce the spherical basis $(\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_z)$:

$$\mathbf{e}_+ = -\frac{1}{\sqrt{2}}\left(\mathbf{e}_x + i\mathbf{e}_y\right) \qquad \mathbf{e}_- = \frac{1}{\sqrt{2}}\left(\mathbf{e}_x - i\mathbf{e}_y\right). \tag{45}$$

The complex projections B_+ and B_- onto this basis are given by the scalar product $\mathbf{e}_\pm^* \cdot \mathbf{B}_1$:

$$B_+ = \frac{1}{2\sqrt{2}}\left(-B_x\, e^{-i\phi_x} + iB_y e^{-i\phi_y}\right) \qquad B_- = \frac{1}{2\sqrt{2}}\left(B_x\, e^{-i\phi_x} + iB_y e^{-i\phi_y}\right).$$

Because of the ε sign in the exponentials, B_+ is the σ^+ component of the rf field for $\varepsilon = 1$, and the σ^- component for $\varepsilon = -1$. We see that $\hat{\mathbf{S}} \cdot \mathbf{e}_+ = -\frac{1}{\sqrt{2}}\hat{S}_+$ and $\hat{\mathbf{S}} \cdot \mathbf{e}_- = \frac{1}{\sqrt{2}}\hat{S}_-$. If we define the complex coupling amplitudes $\Omega_\pm = \mp\sqrt{2} g_s \mu_B B_\pm$, the total spin hamiltonian reads:

$$\hat{H} = \varepsilon\omega_0 \hat{S}_z + \left[\frac{\Omega_+}{2}\, e^{-i\varepsilon\omega t}\, \hat{S}_+ + \frac{\Omega_-}{2}\, e^{-i\varepsilon\omega t}\, \hat{S}_- + h.c.\right]. \tag{46}$$

Let us write $\Omega_+ = |\Omega_+| e^{-i\phi_+}$. After a rotation around z of angle $\phi_+ + \varepsilon\omega t$, and application of the rotating wave approximation, the effective hamiltonian is

$$\hat{H}_{\text{eff}} = -\varepsilon\delta\hat{S}_z + \left[\frac{|\Omega_+|}{2}\,\hat{S}_+ + h.c.\right] = -\varepsilon\delta\hat{S}_z + |\Omega_+|\hat{S}_x. \tag{47}$$

The eigenenergies are $m\hbar\sqrt{\delta^2 + |\Omega_+|^2}$. The eigenstates are again deduced from $|m\rangle_z$ by a rotation of θ around y, where

$$\cos\theta = \frac{-\varepsilon\delta}{\sqrt{\delta^2 + |\Omega_+|^2}} \qquad \sin\theta = \frac{|\Omega_+|}{\sqrt{\delta^2 + |\Omega_+|^2}}. \tag{48}$$

It is clear from the form (47) that the relevant coupling is only the σ^+ polarized part of the rf field for $\varepsilon = 1$ (the σ^- component for $\varepsilon = -1$). It is related to the x and y projections of the rf field through

$$|\Omega_+| = \frac{|g_S|\mu_B}{2\hbar}\sqrt{B_x^2 + B_y^2 + 2B_x B_y \sin(\phi_x - \phi_y)}. \tag{49}$$

[3] In fact, misalignment effect of the rf field, where there is a non zero component along the static field, do have an effect, see [8]. We will discuss this if time allows.

For linearly polarized field, with $B_y = 0$, we recover the amplitude of Eq.(35). The coupling is maximum for purely circular polarization σ^ε, which we obtain when $\phi_x - \phi_y = \pi/2$ and $B_x = B_y$. The amplitude is then twice as large as in the case of the linear field along x. For a linear transverse polarization with $\phi_x = \phi_y$ and $B_x = B_y = B_1$, the coupling is smaller by $\sqrt{2}$ than for the circular case. Finally, the coupling totally vanishes in the case of a $\sigma^{-\varepsilon}$ polarization (σ^- for $\varepsilon = 1$, and vice versa).

We must emphasize that all this reasoning has been done with the direction of the static magnetic field for the quantization axis. If the direction of this field changes in space, the relevant amplitude is the σ^ε component along the new, local direction of the magnetic field.

3.4 Misalignment effects of the rf polarization

We said in the beginning of this section that the z component of the rf field doesn't couple the spin states. In fact, this is not strictly true. When the rf field is has some component along the axis z of the magnetic field, its effect is to modify the Landé factor, by a factor $J_0(\Omega_z/\omega)$ [9]. It can also lead to transitions at submultiples of the Larmor frequency.

Suppose the hamiltonian is a static field along z and a rf field with a z projection writes:

$$\hat{H} = (\omega_0 + \Omega_z \sin \omega t)\ \hat{S}_z + \left\{ \frac{\Omega_+}{2}\, e^{-i\omega t} \hat{S}_+ + h.c. \right\}.$$

The rf field is circularly polarized, but also has a linear component of its polarization along z. What is the effect of this Ω_z term?

To understand is, we will look for the solution of a spin rotated through $R = R_z(\frac{\Omega_z}{\omega} \cos \omega t)$. This rotation is chosen to cancel the Ω_z term of the initial hamiltonian. We get:

$$\hat{H}' = R^\dagger \hat{H} R - \Omega_z \sin \omega t\, \hat{S}_z = \omega_0 \hat{S}_z + \left\{ \frac{\Omega_+}{2}\, e^{-i\omega t} e^{i \frac{\Omega_z}{\omega} \cos \omega t} \hat{S}_+ + h.c. \right\}.$$

The exponential of the cosine may be expanded in terms of Bessel functions of the first kind. This gives:

$$\hat{H}' = \omega_0 \hat{S}_z + \left\{ \frac{\Omega_+}{2} \sum_{n=-\infty}^{+\infty} i^n J_n\left(\frac{\Omega_z}{\omega}\right)\, e^{-i(1+n)\omega t} \hat{S}_+ + h.c. \right\}.$$

Rotations around z by angles $(n+1)\omega t$ will each time make one term in the sum stationary. This means that resonances appear, at frequencies ω such that $(n+1)\omega = \omega_0$, with a coupling amplitude given by the Bessel function:

$$\text{coupling} \quad \Omega_+ J_n\left(\frac{\Omega_z}{\omega}\right) \quad \text{at frequency} \quad \omega = \frac{\omega_0}{n+1}.$$

The $n = 0$ case is the usual, expected transition. However, the rf coupling is modified and is now $\Omega_+ J_0\left(\frac{\Omega_z}{\omega}\right)$. We recover a coupling Ω_+ when Ω_z vanishes. Everything happens as if the Landé factor had been modified [9] by a factor $J_0\left(\frac{\Omega_z}{\omega}\right)$, smaller than one, which can even change sign if Ω_z is comparable with ω.

The cases $n > 0$ correspond to resonances at submultiples of the Larmor frequencies [8], with smaller amplitudes $\Omega_+ J_n\left(\frac{\Omega_z}{\omega}\right)$. For $\Omega_z \ll \omega$, the coupling amplitude scales as $\left(\frac{\Omega_z}{\omega}\right)^n$

and is very small. For practical purposes in rf-dressed adiabatic potentials, the rf source often has harmonics of the frequency ω due to non linear amplification. This misalignment effect is another reason for avoiding to have atoms at a position when the Larmor frequency is close to 2ω. For the rest of the lecture, we will ignore the effect of the Ω_z term, and take it equal to zero.

4 The dressed state picture

4.1 Field quantization

Although the rf field is classical in the sense that the mean photon number $\langle N\rangle$ interacting with the atoms is very large, and its relative fluctuations $\Delta N/\langle N\rangle$ negligible, it gives a deeper insight in the coupling to use a quantized description for the rf field. This will make much clearer the interpretation of rf spectroscopy or the effect of strong rf coupling, beyond RWA. We will chose $\varepsilon = +1$ for simplicity, the other choice simply changing the role of the two polarizations.

We start from the expression of the classical field in the spherical basis:

$$\mathbf{B}_1 = B_+\, e^{-i\omega t}\mathbf{e}_+ + B_-\, e^{-i\omega t}\mathbf{e}_- + c.c.$$

The quantum rf magnetic field can be described as follows:

$$\hat{\mathbf{B}}_1 = (b_+\,\mathbf{e}_+ + b_-\,\mathbf{e}_-)\; a + h.c. \tag{50}$$

where $b_\pm = B_\pm/\sqrt{\langle N\rangle}$. Defining the Rabi coupling as $\Omega_\pm = \mp\sqrt{2}g_s\mu_B B_\pm$, and the one-photon Rabi coupling as $\Omega_\pm^{(0)} = \mp\sqrt{2}g_s\mu_B b_\pm = \Omega_\pm/\sqrt{\langle N\rangle}$, the total hamiltonian for the spin and the field reads:[4]

$$\hat{H} = \omega_0\hat{S}_z + \hbar\omega a^\dagger a + \left[\frac{\Omega_+}{2\sqrt{\langle N\rangle}}\left(a\,\hat{S}_+ + a^\dagger\,\hat{S}_-\right) + \frac{\Omega_-}{2\sqrt{\langle N\rangle}}\left(a\,\hat{S}_- + a^\dagger\,\hat{S}_+\right)\right]. \tag{51}$$

4.2 Uncoupled states

In the absence of coupling (for $\Omega_\pm = 0$), the eigenstates of the { atom + photons } system $\hat{H}_0 = \omega_0\hat{S}_z + \hbar\omega a^\dagger a$ are $|m,N\rangle = |m\rangle_z|N\rangle$, where $|m\rangle_z$ is an eigenstate of $\hat{S}_z$ and $|N\rangle$ an eigenstate of $a^\dagger a$, with respective eigenvalues $m\hbar$ and N:

$$\hat{H}_0|m,N\rangle = E^0_{m,N}|m,N\rangle, \qquad E_{m,N} = m\hbar\omega_0 + N\hbar\omega.$$

Let us write this energy in terms of the detuning $\delta = \omega - \omega_0$:

$$E_{m,N} = -m\hbar\delta + (N+m)\hbar\omega.$$

From this expression, we see that the states in the manifold $\mathcal{E}_N = \{|m,N-m\rangle, m = -S\ldots S\}$ have an energy

$$E_{m,N-m} = -m\hbar\delta + N\hbar\omega.$$

For a rf frequency ω close to ω_0, that is if $\delta \ll \omega$, the energy splitting inside a manifold, of order $\hbar\delta$, is very small as compared to the energy splitting between different manifolds, which is $\hbar\omega$, see Fig. 3.

[4] We set the origin of energy so as to include the zero photon energy $\hbar\omega/2$.

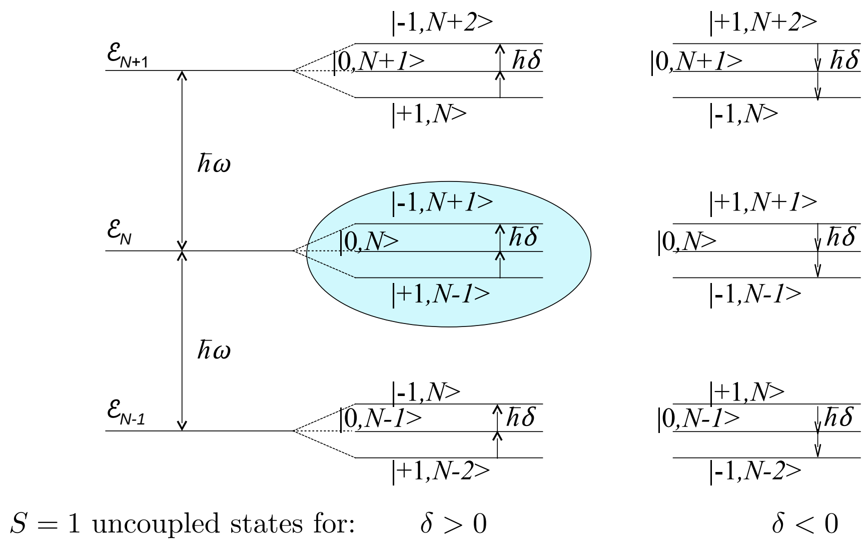

Figure 3: The unperturbed atom + field states can be grouped into manifolds of $2S+1$ states with a small energy difference $\hbar\delta$ compared to the energy spacing between manifolds $\hbar\omega$. Depending on the sign of δ, either the state connected to $|+S\rangle_z$ or to $|-S\rangle_z$ has a larger energy. On resonance ($\delta = 0$), all the states are degenerate. The $\mathcal{E}_N$ manifold with mean energy $N\hbar\omega$ and $\delta > 0$ is enlightened.

4.3 Effect of the rf coupling

There are four coupling terms, illustrated on Fig. 4. The two first terms, proportional to Ω_+, act inside a given $\mathcal{E}_N$ manifold:

$$\left\langle m \pm 1, N - m \mp 1 \left| \frac{\Omega_+}{2\sqrt{\langle N \rangle}} \left(a\, \hat{S}_+ + a^\dagger\, \hat{S}_- \right) \right| m, N - m \right\rangle \simeq \frac{\Omega_+}{2} \sqrt{S(S+1) - m(m \pm 1)}.$$

As $\langle N \rangle \ll 1$, we have neglected the difference between $N - m$ and $\langle N \rangle$ when applying a and $a^\dagger$.

The two last terms, proportional to Ω_-, couple states of the $\mathcal{E}_N$ manifold to states of the $\mathcal{E}_{N\pm 2}$ manifold:

$$\left\langle m \pm 1, N - m \pm 1 \left| \frac{\Omega_-}{2\sqrt{\langle N \rangle}} \left(a^\dagger\, \hat{S}_+ + a\, \hat{S}_- \right) \right| m, N - m \right\rangle \simeq \frac{\Omega_-}{2} \sqrt{S(S+1) - m(m \pm 1)}.$$

An estimation of the effect of this two terms on the energy with perturbation theory with lead to a shift of order $\hbar|\Omega_+|^2/\delta$ for the Ω_+ terms, and $\hbar|\Omega_-|^2/(\omega_0 + \omega)$ for the Ω_- term. The rotating wave approximation, which applies if $|\delta|, |\Omega_\pm| \ll \omega$, consists in neglecting the effect of the Ω_- terms, and to concentrate on a given multiplicity.

4.4 Dressed states in the rotating wave approximation

Within the rotating wave approximation, we can just find the eigenstates in a given manifold. The result is given by using generalized spin rotation, with the angle given at Eq.(48),

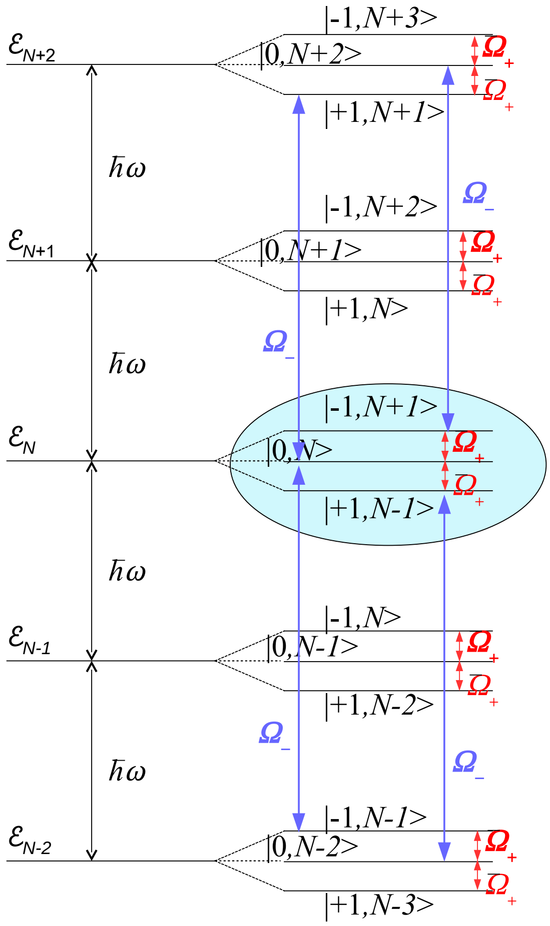

Figure 4: Coupling terms between unperturbed states. The Ω_+ terms couple states inside a given $\mathcal{E}_N$ manifold, whereas the Ω_- terms couple states from different $\mathcal{E}_N$ manifolds.

changing the photon number accordingly to stay in the $\mathcal{E}_N$ manifold. The eigenstates are the *dressed states* $|m', N\rangle$ with energies

$$E'_{m'} = m'\hbar\sqrt{\delta^2 + |\Omega_+|^2}.$$

The spin states are dressed by the rf field, in such a way that the eigenstates are now combining different spin and field states, and cannot be written as a product spin$\otimes$field. The dressed states are connected to the uncoupled states for $|\delta| \gg 1$. The effect of the coupling is to repel the states inside the multiplicity, the frequency splitting going from $|\delta|$ to $\sqrt{\delta^2 + |\Omega_+|^2}$.

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