Low-dimensional Bose gases
Part 1: BEC and interactions

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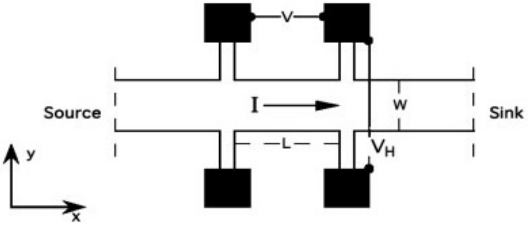
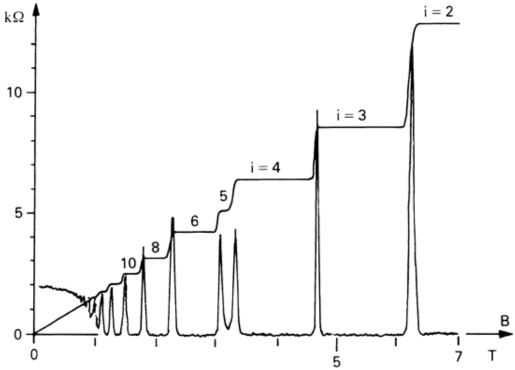
Physics is qualitatively changed when dimension is reduced.
Examples include:

- in 1D: absence of thermalisation of a 1D gas, ‘fermionization’ of an interacting Bose gas, renormalization of the interactions...
- in 2D: (fractional) quantum Hall effect, Kosterlitz-Thouless transition, renormalization of the interactions...

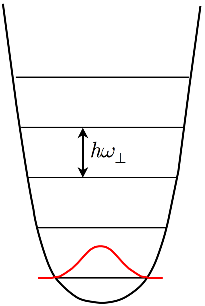
- 2D electron gas at the interface of a semiconductor heterojunction
- longitudinal current I_x, high perpendicular magnetic field B_z
- measure the transverse voltage $V_H = V_y$





- plateaux of Hall resistance $R = \frac{V_y}{I_x} = \frac{h}{i e^2}, \quad i \in \mathbb{N}^*$
- longitudinal resistance $R_x = \frac{V_x}{I_x} = 0$

Experimental realization of low-D gases: strongly confine $3 - D$ directions ($k_B T, \mu \ll \hbar\omega_\perp$)



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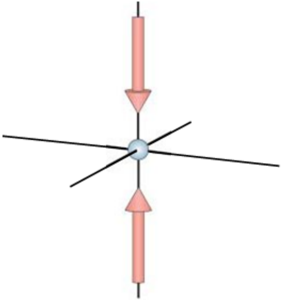

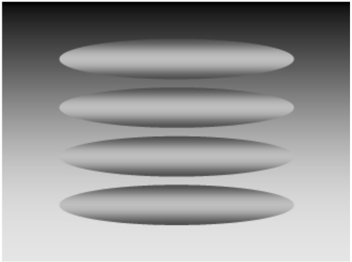

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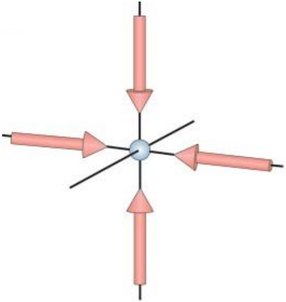

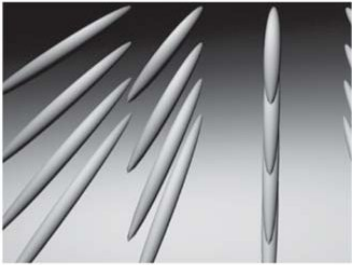

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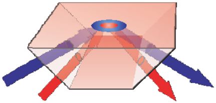

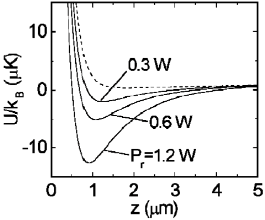

Innsbruck

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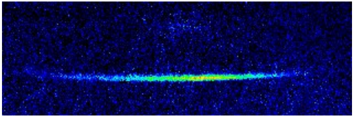
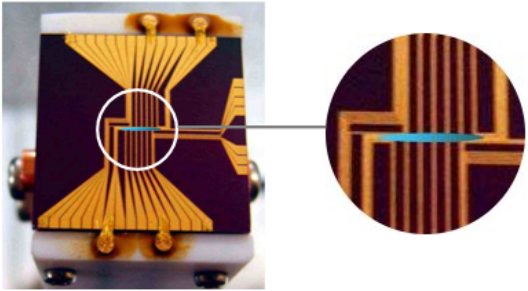
Villetaneuse 

Experimental realization of low-D gases: strongly confine $3-D$ directions ($k_B T, \mu \ll \hbar\omega_\perp$)

- optical lattices in $3-D$ directions
- 2D optical surface traps / rf-dressed magnetic traps
- anisotropic magnetic traps on chips



Many groups...
including Vienna!

Introduction

General references:

- *Bose-Einstein Condensation*, Lev Pitaevskii and Sandro Stringari, Oxford (2003)
- *Quantum Gases in Low Dimensions*, edited by L. Pricoupenko, H. Perrin and M. Olshanii, J. Phys IV **116** (2004)
  Les Houches lectures by Shlyapnikov, Castin, Olshanii, Stringari, Cirac and Douçot.
- *Many body physics with ultra cold gases*, I. Bloch, J. Dalibard and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008)

... and (many) references therein.

Outline

Outline of the lecture

BEC of an ideal gas in reduced dimensions

Reminder: Bose-Einstein condensation

non interacting Bose gas, non degenerate ground state of energy ε_0
semi-classical approach (valid if $k_B T \gg \Delta\varepsilon \sim \hbar\omega_0$ or $\hbar^2/2ML^2$)
Bose-Einstein distribution in the grand canonical ensemble:

$$n(\varepsilon) = \frac{1}{\exp(\beta(\varepsilon - \mu)) - 1} \geq 0$$

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$\rho(\varepsilon)$: density of states, depends on the system (trap, free bosons...)
$N_0 = n(\varepsilon_0)$: mean number of particles in the ground state
N': mean number of particles in the excited states

Reminder: Bose-Einstein condensation

The occupancy $n'(\varepsilon)$ of each excited level is bounded from above using $\mu < \varepsilon_0$:

$$n'(\varepsilon) < \frac{1}{\exp(\beta(\varepsilon - \varepsilon_0)) - 1} = \sum_{n=1}^{\infty} e^{-n\beta(\varepsilon - \varepsilon_0)}.$$

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If this integral is finite, $N_0 > N - N_C(T)$.
T_C such that $N_C(T_C) = N$. $N_C(T)$ increases with T.
$T < T_C \Longrightarrow N_0 > 0$ Bose-Einstein condensation

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$T < T_C \Longrightarrow N_0 > 0$ Bose-Einstein condensation
Depending on $\rho(\varepsilon)$, N_C is finite or not...

Role of the density of states

An important particular case: power law density of state
$\rho(\varepsilon) \propto (\varepsilon - \varepsilon_0)^k$ with $\varepsilon > \varepsilon_0$

$$N_C(T) \propto \sum_{n=1}^{\infty} \int_0^{\infty} \varepsilon^k e^{-n\beta\varepsilon} d\varepsilon \quad \propto \quad (k_B T)^{k+1} \sum_{n=1}^{\infty} \frac{1}{n^{k+1}}$$

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Converges for $k > 0$.

Fraction of condensed particles:

$$N_C(T_C) = N \quad \Longrightarrow \quad \frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^{k+1}$$

A slightly different approach

- at fixed T, calculate $N'(\mu) = \int \rho(\varepsilon) n'(\varepsilon) d\varepsilon$
- For $\rho(\varepsilon) \propto (\varepsilon - \varepsilon_0)^k$, $N'(z) \propto T^{k+1} g_{k+1}(z)$
  as a function of the fugacity $z = e^{\beta(\mu - \varepsilon_0)} < 1$.

$$g_{k+1}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{k+1}} \text{ polylogarithm or Bose function.}$$

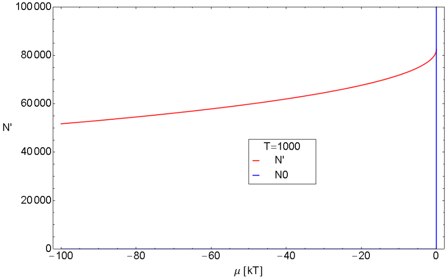
  increasing function of z (or μ).
- If N' can take any value for $\mu < \varepsilon_0$ or $z < 1$, no BEC. For any N, one can find a μ such that $N'(\mu) = N$ and $N_0 \ll N$.
- If $N'(\mu) = N$ has no solution for large enough N, Bose-Einstein condensation.
- Again, $g_{k+1}(1)$ is finite for $k < 0$.

Example: Bose-Einstein condensation in a 3D box

3D box: $\rho(\varepsilon) \propto \sqrt{\varepsilon}, k = \frac{1}{2} \Rightarrow$ series $\sim \frac{1}{n^{3/2}}$ converges,
$g_{3/2}(1) = 2.612$ is finite

$$N'(T, \mu) = \frac{L^3}{\lambda^3} g_{3/2}(e^{\beta\mu})$$

where $\lambda = \dfrac{h}{\sqrt{2\pi M k_B T}}$



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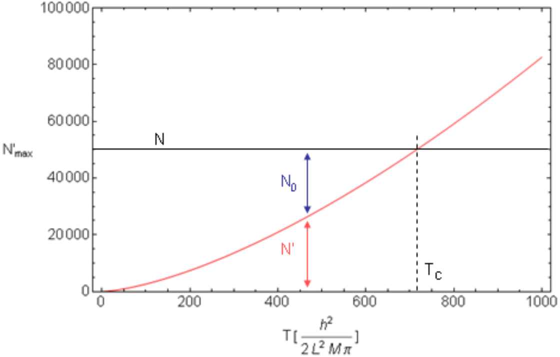
$$N_C(T) = 2.612 \frac{L^3}{\lambda^3}$$

$$N_C(T) \propto T^{3/2}$$



Saturation of the excited states: T_C given by $\frac{N}{L^3}\lambda^3 = 2.612$

BEC in lower dimensions?

Does BEC also happen in lower dimensions?

- box of dimension D:
  $\rho(\varepsilon) \propto \varepsilon^{\frac{D}{2}-1} \quad \Longrightarrow \quad D > 2$
  BEC possible only for $D = 3$ at finite T in thermodynamic limit

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$$k_B T_C \sim N^{\frac{1}{D}} \hbar\omega_0 \qquad \frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^D$$

N.B. $T_C \sim T_d$ degeneracy temperature $k_B T_d = N^{\frac{1}{D}} \hbar\omega_0$

1D harmonic trap

Finite size effect: refining the 1D trapped case
gap $\Delta\varepsilon = \hbar\omega_0$ to the first excited state

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$$N_C(T) = \sum_{n=1}^{\infty} \int_{\hbar\omega_0}^{\infty} \rho(\varepsilon) e^{-n\beta\varepsilon} d\varepsilon = \frac{k_B T}{\hbar\omega_0} \sum_{n=1}^{\infty} \frac{e^{-n\beta\hbar\omega_0}}{n}$$

$$N_C(T) = -\frac{k_B T}{\hbar\omega_0} \mathrm{Ln}(1 - e^{-\beta\hbar\omega_0}) \simeq \frac{k_B T}{\hbar\omega_0} \mathrm{Ln}\left(\frac{k_B T}{\hbar\omega_0}\right)$$

$$k_B T_C \simeq \hbar\omega_0 \frac{N}{\mathrm{Ln} N}$$

N.B. $T_C \ll T_d$ with $T_d = N\hbar\omega_0$ degeneracy temperature;
$\frac{T_C}{T_d} \longrightarrow 0$ in the thermodynamic limit ($\frac{T_C}{T_d} = $ cst in 2D or 3D)

Coherence of the condensate

coherence is described by first order correlation function

$$g^{(1)}(\mathbf{r},\mathbf{r}') = \frac{\langle\hat{\psi}^{+}(\mathbf{r})\hat{\psi}(\mathbf{r}')\rangle}{\sqrt{n(\mathbf{r})n(\mathbf{r}')}}$$

Above T_C: Gaussian decay: $g^{(1)}(\delta r) = e^{-\pi\frac{\delta r^2}{\lambda^2}}$ box / trap

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Below T_C: phase fluctuations are dominant $\hat{\psi}(\mathbf{r}) = \sqrt{n(\mathbf{r})}e^{i\hat{\phi}(\mathbf{r})}$

$$g^{(1)}(\mathbf{r},\mathbf{r}') \simeq \langle e^{i(\hat{\phi}(\mathbf{r})-\hat{\phi}(\mathbf{r}'))}\rangle = e^{-\frac{1}{2}\langle\delta\hat{\phi}^2\rangle}$$

phase fluctuations $\delta\hat{\phi} = \phi(\mathbf{r}) - \phi(\mathbf{r}')$ determine the coherence of the Bose gas.

Coherence of the condensate

phase fluctuations are increased in reduced dimension.
In the limit $\delta r \to \infty$:

- 3D: $\langle \delta\hat{\phi}^2 \rangle \sim \text{cst} \Rightarrow g^{(1)} \sim \frac{N_0}{N}$ long range order

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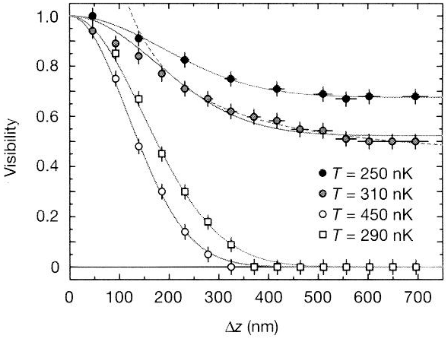

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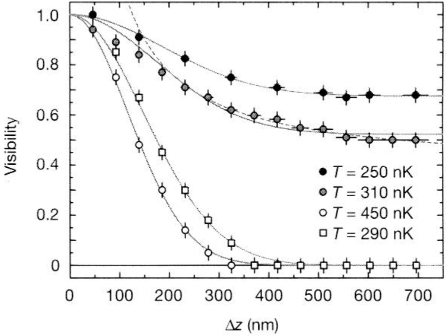

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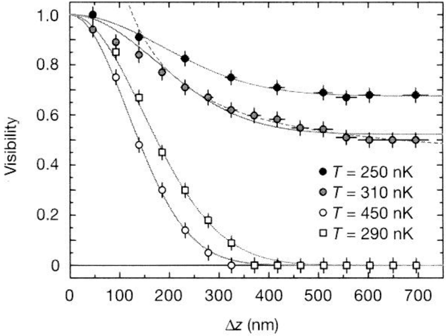

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- 2D: $\langle \delta\hat{\phi}^2 \rangle \sim \ln(\delta r) \Rightarrow g^{(1)} \sim \delta r^{-\frac{1}{n\lambda^2}}$ algebraic decay
- 1D: $\langle \delta\hat{\phi}^2 \rangle \sim \delta r \Rightarrow g^{(1)} \sim e^{-\frac{\delta r}{\ell_\phi}}$ exponential decay

Coherence of the condensate

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phase domains of size $\ell_\phi \propto \frac{T_\phi}{T}$

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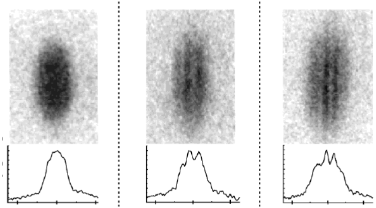
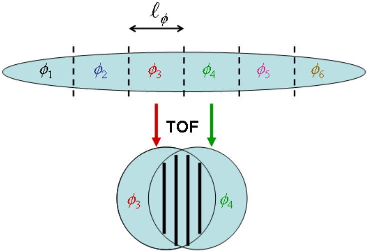
$T = 0.9\,T_C$





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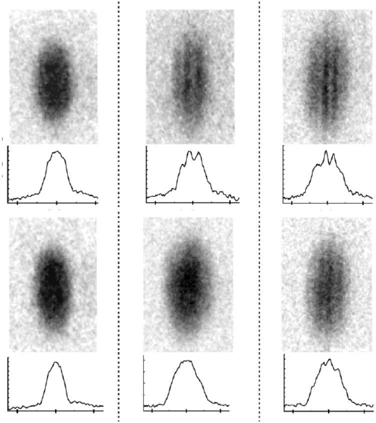

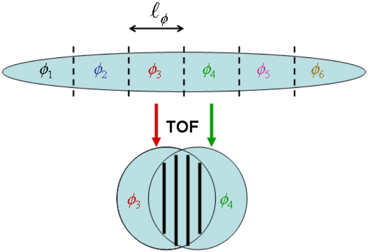

Interactions in lower dimension

Scattering theory

Rigorous approach: solve the scattering problem in dimension D.

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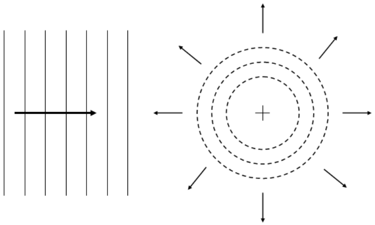
Reminder: in 3D, s-wave scattering at low energy $\Rightarrow$ simple dephasing $-ka$ of the wave function



plane wave e^{ikz} spherical wave $\frac{e^{ik(r-a)}}{r}$

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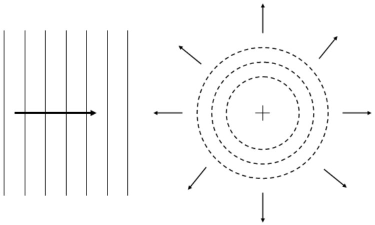


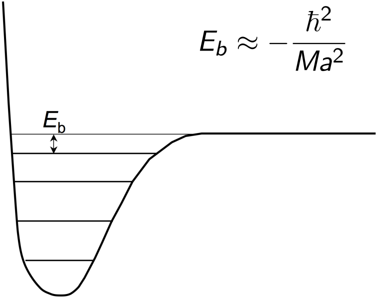
plane wave e^{ikz} spherical wave $\frac{e^{ik(r-a)}}{r}$

scattering length a contains all relevant scattering information
$\Rightarrow$ use an effective contact interaction $g\delta(\mathbf{r})$

$$g_{3D} = \frac{4\pi\hbar^2 a}{M}$$

Formation of molecules

- $a \to \infty$ if there is a molecular bound state of zero energy
- $a > 0$ and large: last bound state close to dissociation threshold



$$E_b \approx -\frac{\hbar^2}{Ma^2}$$

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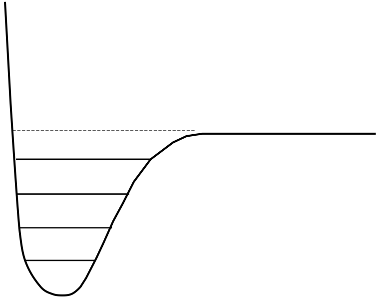

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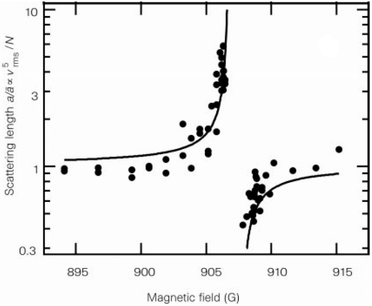
Feshbach resonance in sodium
Inouye et al. (1998)



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- At a Feshbach resonance, the scattering length diverges and changes sign when varying B.
- A B ramp from $a < 0$ to $a > 0$ can produce molecules with binding energy $E_b = -\dfrac{\hbar^2}{Ma^2}$.

Gross-Pitaevskii equation

dilute gas ($na^3 \ll 1$); all particles in the same single particle state; mean field approach; g enters in the interaction term of Gross-Pitaevskii equation for the condensate wave function

$$-\frac{\hbar^2}{2M}\triangle\psi + U(\mathbf{r})\psi + g|\psi|^2\psi = \mu\psi$$

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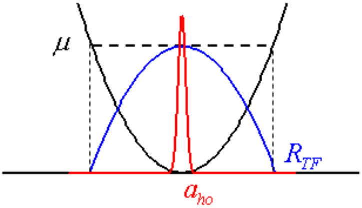
Thomas Fermi regime if $Na \gg a_{\text{ho}}$ or $\mu \gg \hbar\omega$:

$$|\psi|^2 = n(\mathbf{r}) = \frac{\mu - U(\mathbf{r})}{g}$$



in a harmonic trap:

$$\mu \propto (Na)^{2/5}$$

$$R_{\text{TF}} \propto (Na)^{1/5}$$

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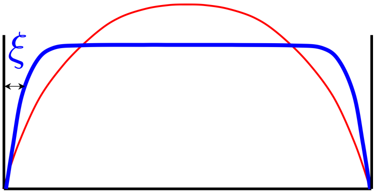
Thomas Fermi regime in a box: uniform n except at the edges, on a size ξ



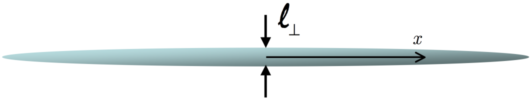
healing length ξ:

$$\frac{\hbar^2}{2M\xi^2} = \mu$$

$$\xi = \frac{1}{\sqrt{8\pi na}}$$

Interactions in dimension D

What about interactions in dimension D?



$3 - D$ directions confined to the ground state of an harmonic oscillator $\omega_\perp$, to a size $\ell_\perp = \sqrt{\hbar/M\omega_\perp}$. Implies $\mu_D \ll \hbar\omega_\perp$.

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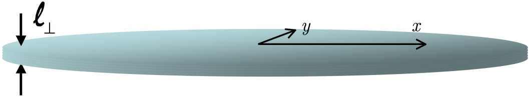

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Interactions in dimension D

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Two situations:

- $\ell_\perp > a$
  (thermo)dynamics in dimension D, collisions still in 3D
- $\ell_\perp < a$
  (thermo)dynamics and collisions in dimension D

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Typically, $\ell_\perp > 30\,\mathrm{nm}$, $a \sim$ a few nm $\Rightarrow \ell_\perp > a$...
... unless a Feshbach resonance is used

Regular case: $\ell_\perp > a$

case $\ell_\perp > a$: write $\psi(\mathbf{r}) = \psi_D(\mathbf{r}_D)\phi_\perp(\mathbf{r}_\perp)$ and look for a GPE in dimension D:

$$-\frac{\hbar^2}{2M}\triangle_D\psi_D + U(\mathbf{r}_D)\psi_D + g_D|\psi_D|^2\psi_D = \mu_D\psi_D$$

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deduce g_D from averaging the interaction over the transverse distribution $n_\perp(\mathbf{r}_\perp) = |\phi_\perp(\mathbf{r}_\perp)|^2$

$$g_D|\psi_D(\mathbf{r}_D)|^2 = \int g|\psi|^2|\phi_\perp(\mathbf{r}_\perp)|^2 d\mathbf{r}_\perp = g|\psi_D(\mathbf{r}_D)|^2\int|\phi_\perp(\mathbf{r}_\perp)|^4 d\mathbf{r}_\perp$$

$$g_D = \frac{g}{(\sqrt{2\pi}\ell_\perp)^{3-D}}$$

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- 1D $\quad g_1 = 2\hbar\omega_\perp a$
- 2D $\quad g_2 = \frac{\hbar^2}{M}\frac{\sqrt{8\pi}a}{\ell_\perp} = \frac{\hbar^2}{M}\tilde{g}_2$

Exotic case: $\ell_\perp < a$

case $\ell_\perp < a$: renormalization of the interaction constant

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case $\ell_\perp < a$: renormalization of the interaction constant

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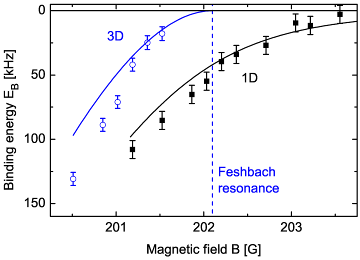
$$g_{1D} = \frac{g_1}{1 - A\frac{a}{\ell_\perp}}, \quad g_1 = 2\hbar\omega_\perp a, \quad A \simeq 1$$

diverges for $a \sim \ell_\perp \Rightarrow$ confinement-induced resonance



Experiment with fermions:
confinement-induced bound state of ^{40}K (Moritz et al., 2005)
molecular bound state even for $a < 0$
when $\ell_\perp \sim |a|$

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case $\ell_\perp < a$: renormalization of the interaction constant

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- 2D

$$g_{2D} = \frac{g_2}{1 + \frac{a}{\sqrt{2\pi}\ell_\perp}\ln(B/\pi k^2\ell_\perp^2)}, \quad g_2 = \frac{\hbar^2}{M}\frac{\sqrt{8\pi}a}{\ell_\perp}, \quad B \simeq 0.9$$

  - $g_{2D} > 0$ for small $\ell_\perp$ even if $a < 0$
  - $k^2 \sim \frac{M}{\hbar^2}\mu \sim \frac{M}{\hbar^2}g_2 n = \tilde{g}_2 n \Rightarrow g_{2D}$ depends on atomic density

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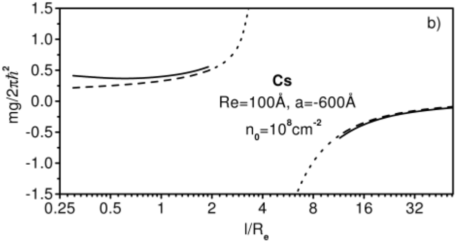
- 2D

Petrov, Holzmann, Shlyapnikov (2000)



confinement-induced resonance for $a < 0$

Strong or weak interaction?

Compare interaction energy $E_I = ng_D$ to kinetic energy E_K to localize particles within $\ell = n^{-1/D}$ in dimension D

$$E_K \sim \frac{\hbar^2}{M\ell^2} = \frac{\hbar^2 n^{2/D}}{M} \quad \Longrightarrow \quad \frac{E_I}{E_K} \sim \frac{Mg_D}{\hbar^2} n^{\frac{D-2}{D}}$$

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- 3D: $\frac{E_I}{E_K} \sim 4\pi(na^3)^{1/3}$
  gas parameter; strong interaction at high density
- 1D: $\frac{E_I}{E_K} \sim \frac{Mg_1}{\hbar^2 n} \sim \frac{2a}{n\ell_\perp^2} = \gamma$
  N.B. strong interaction $\gamma \gg 1$ means low density!

Strong or weak interaction?

2D interacting gas: $\frac{E_I}{E_K} \sim \frac{Mg_D}{\hbar^2} n^{\frac{D-2}{D}} = \frac{Mg_{2D}}{\hbar^2}$

- $\frac{M}{\hbar^2} g_{2D} = \frac{\tilde{g}_2}{1 + \frac{a}{\sqrt{2\pi}\ell_\perp} \ln(B/\pi k^2 \ell_\perp^2)}$, $\quad k \sim \sqrt{n}$, $\tilde{g}_2 = \frac{\sqrt{8\pi} a}{\ell_\perp}$

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- $a > \ell_\perp \Longrightarrow a_{2D} \simeq \sqrt{\frac{\pi}{B}} \ell_\perp$ and $\dfrac{M}{\hbar^2} g_{2D} \simeq \dfrac{4\pi}{\ln(1/n a_{2D}^2)}$

  weak interaction for $\frac{1}{\ln(1/n a_{2D}^2)} \ll 1$ 2D gas parameter

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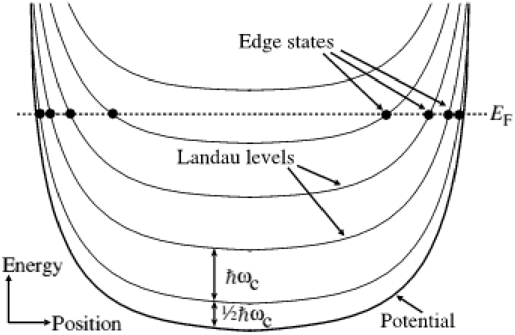
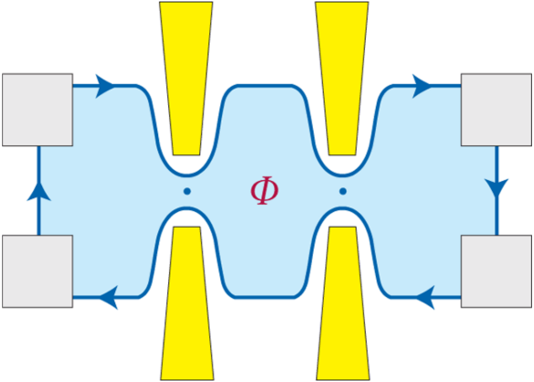
  weak interaction for $\frac{1}{\ln(1/na_{2D}^2)} \ll 1$ 2D gas parameter
- $a \ll \ell_\perp \implies \dfrac{M}{\hbar^2} g_{2D} \simeq \tilde{g}_2$; density independent criterion

- Hall plateau for a chemical potential between Landau levels
- conductivity restricted to the edges = 1D channels



Introduction
Example in 2D: the Quantum Hall Effect



- edge states = 1D systems
- can carry excitations of fractional charge