Atoms and photons

Illustrations for Chapter 3

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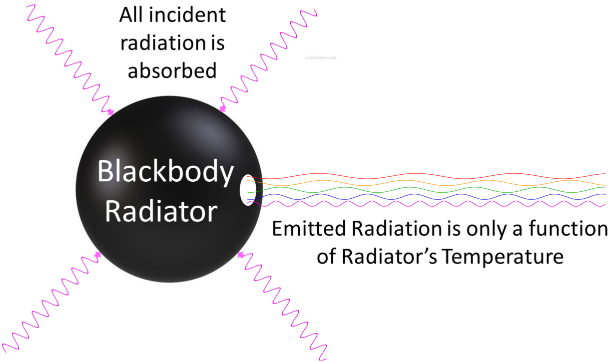
October 26, 2022

The Blackbody problem

Emission by a small hole in a heated oven.



The Blackbody problem

Emission by a small hole in a heated oven. What is known at Planck's time:

- The radiation is universal
- Stefan's law

$$\mathcal{P} = \sigma S T^4 \tag{1}$$

where $\sigma = 5.67\,10^{-8}$ W/m^{2}K^4

- Lambert's law

$$d\mathcal{P} = LS\cos\theta\, d\Omega \tag{2}$$

where the luminance L is related to the total density of energy in the oven $u = \int u_\nu\, d\nu$, by:

$$L = \frac{cu}{4\pi} \Rightarrow \mathcal{P} = \frac{cSu}{4} \tag{3}$$

and

$$u = \frac{4}{c}\sigma T^4 \tag{4}$$

The Blackbody problem

Emission by a small hole in a heated oven. What is known at Planck's time:

- Wien's displacement law

$$u_\nu = \nu^3 f\left(\frac{\nu}{T}\right) \quad (5)$$

- Wien's phenomenological model at high frequencies

$$u_v = \alpha\nu^3 e^{-\gamma\nu/T} \ , \quad (6)$$

- And many precise measurements of the spectrum (pyrometry).

Problem: how can we derive the law for any frequency? Can we compute σ from physical constants?

The Blackbody problem

Counting the modes

Assume a cubic volume $\mathcal{V} = L^3$ for the oven, with periodic boundary conditions. Support only plane waves with $\mathrm{k} = (k_x, k_y, k_z)$ so that

$$k_x = \frac{2\pi}{L} n_x \tag{7}$$

where $n_{x,y,z}$ is a set of three positive or negative integers. Two orthogonal polarizations for each set of integers. Energies of all these 'modes' add up independently (detailed justification later).
N_ν the total number of modes $k < 2\pi\nu/c$. Number of modes per unit volume between ν and $\nu + d\nu$: $\rho_\nu \, d\nu$

$$\rho_\nu = \frac{1}{\mathcal{V}} \frac{dN_\nu}{d\nu} \tag{8}$$

The Blackbody problem

Counting the modes

$k < 2\pi\nu/c \Leftrightarrow 2\pi|\vec{n}|/L < 2\pi\nu/c \Leftrightarrow |\vec{n}| < \nu L/c$. Counting the modes with a frequency lower than ν amounts to counting twice (two polarizations) the number of points with integer coordinates in a sphere of radius $\nu L/c$:

$$N_\nu = 2 \times \frac{4\pi}{3}\left(\frac{\nu L}{c}\right)^3 = \frac{8\pi}{3}\frac{\nu^3}{c^3}\mathcal{V}. \tag{9}$$

Hence

$$\rho_\nu = \frac{1}{\mathcal{V}}\frac{dN_\nu}{d\nu} = \frac{8\pi}{c^3}\nu^2. \tag{10}$$

The Blackbody problem

Rayleigh Jeans argument

Attribute the average thermal energy $k_b T$ to each mode

$$u_\nu = k_b T \rho_\nu = \frac{8\pi}{c^3}\nu^2 k_b T \tag{11}$$

- Fits with observation at low frequency
- Absurd at high frequencies: divergence of the spectrum and infinite power

Classical statistical physics fails at explaining the blackbody radiation !

The Blackbody problem

Planck's argument

The light quantum

Planck's hypothesis

The exchanges of energy between field and matter occur as multiples of a fundamental quantum

$$h\nu \tag{12}$$

where h is a 'Hilfeconstant'. Hence $E = nh\nu$.

Average energy per mode (standard statistical physics)

$$\overline{E} = h\nu \frac{\sum_{n=0}^{\infty} ne^{-nh\nu/k_b T}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_b T}} \tag{13}$$

The Blackbody problem

Planck's argument

$$\overline{E} = h\nu \frac{\sum_{n=0}^{\infty} n e^{-nh\nu/k_b T}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_b T}}$$

With $\chi = h\nu/k_b T$, we note that

$$\sum_{n=0}^{\infty} e^{-n\chi} = \frac{1}{1-e^{-\chi}}$$

and

$$\sum_{n=0}^{\infty} n\, e^{-n\chi} = -\frac{d}{d\chi}\frac{1}{1-e^{-\chi}} = \frac{e^{-\chi}}{(1-e^{-\chi})^2}.$$

Finally,

$$\overline{E} = h\nu\overline{n} = h\nu\frac{1}{e^{\chi}-1} \tag{14}$$

The Blackbody problem

Planck's argument

$$\overline{E} = h\nu\bar{n} = h\nu\frac{1}{e^{\chi} - 1}$$

We finally get the Planck's law:

$$u_\nu = \overline{E}\rho_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_b T} - 1} \tag{15}$$

In excellent agreement with experiments if

$$h = 6.62\,10^{-34}\ \mathrm{J}\cdot\mathrm{s} \tag{16}$$

N.B. In terms of $\lambda = c/\nu$, we have

$$u_\lambda = \left|\frac{d\nu}{d\lambda}\right| u_\nu = \frac{c}{\lambda^2}u_\nu = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_b T} - 1}.$$

The Blackbody problem

Limits

- For small frequencies: Rayleigh Jeans

$$u_\nu = \frac{8\pi\nu^2}{c^3} k_b T \tag{17}$$

the classical predictions without field quantization (many photons per mode).

- For large frequencies: phenomenological Wien's law

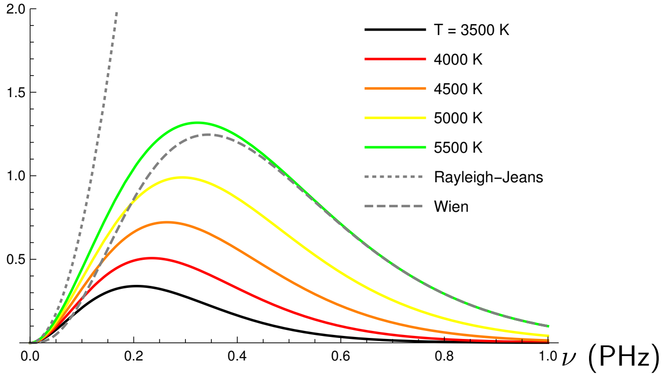
$$u_\nu = \frac{8\pi h\nu^3}{c^3} e^{-h\nu/k_b T} \tag{18}$$

- Explicit expression of Stefan's constant

$$\sigma = \frac{2\pi^5}{15} \frac{k_b^4}{c^2 h^3} \tag{19}$$

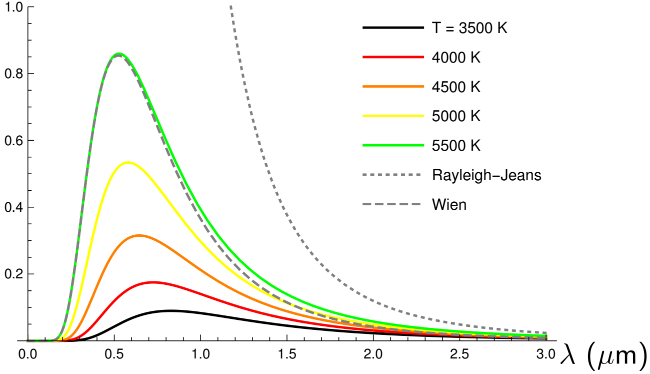
The Blackbody problem

$u_\nu(\nu)$ [J·m^{-3}·PHz^{-1}]



The Blackbody problem

$u_\lambda(\lambda)$ [J·m^{-3} · μm^{-1}]



The Blackbody problem

Einstein 1905, high frequency limit

A more solid justification of the heuristic Plank's hypothesis. Starting point

$$u_\nu = \alpha\nu^3 e^{-h\nu/k_b T} = \alpha\nu^3 e^{-\gamma\nu/T} \tag{20}$$

with $\gamma = h/k_b$. This leads by a simple inversion to:

$$T = -\frac{\gamma\nu}{\ln[u_\nu/\alpha\nu^3]} \tag{21}$$

Density of entropy s, $du = T\,ds$ or $ds/du = 1/T$ and, by integration over u

$$\begin{aligned} s &= -\int_0^u du' \, \frac{\ln[u'/\alpha\nu^3]}{\gamma\nu} \\ &= -\frac{u}{\gamma\nu}\left[\ln\frac{u}{\alpha\nu^3} - 1\right] \end{aligned} \tag{22}$$

The Blackbody problem
Einstein 1905

Total entropy in volume $\mathcal{V}$, $S = s\mathcal{V}$, and total energy $E = u\mathcal{V}$ linked by

$$S = -\frac{E}{\gamma\nu}\left[\ln\frac{E}{\mathcal{V}\alpha\nu^3} - 1\right] \tag{23}$$

S_0 the entropy for the volume $\mathcal{V}_0$

$$S - S_0 = \frac{E}{\gamma\nu}\ln\frac{\mathcal{V}}{\mathcal{V}_0} = k_b\frac{E}{h\nu}\ln\frac{\mathcal{V}}{\mathcal{V}_0} \tag{24}$$

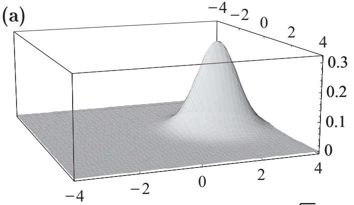
Compare to the entropy variation of a perfect gas in an isothermal compression

$$S - S_0 = k_b N \ln\frac{\mathcal{V}}{\mathcal{V}_0} \tag{25}$$

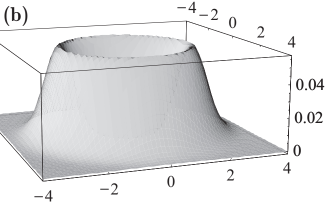
where N is the total number of particles. $N = E/h\nu$ and $E/N = h\nu$.

Husimi-Q function

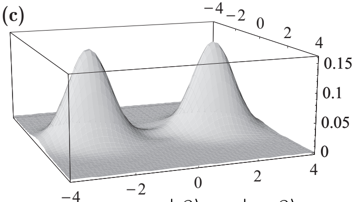
Coherent state, Fock state, cat state and mixture five 5 photons on average



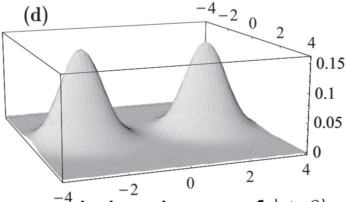
coherent state $\beta = \sqrt{5}$



Fock state $n = 5$



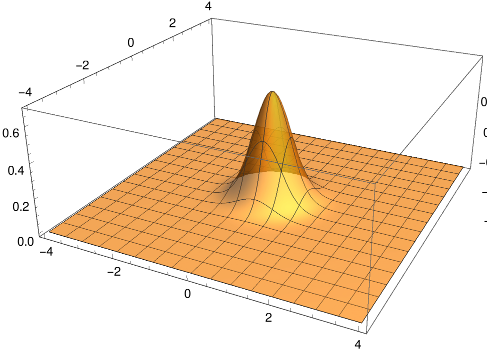
cat state $|\beta\rangle + |-\beta\rangle$



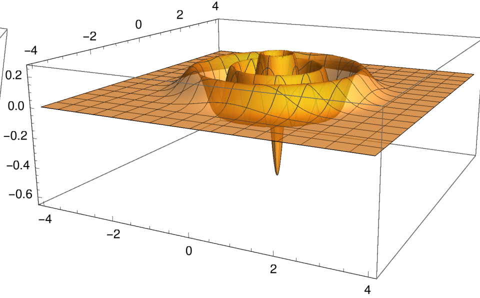
statistic mixture of $|\pm\beta\rangle$

Wigner function

Fock state



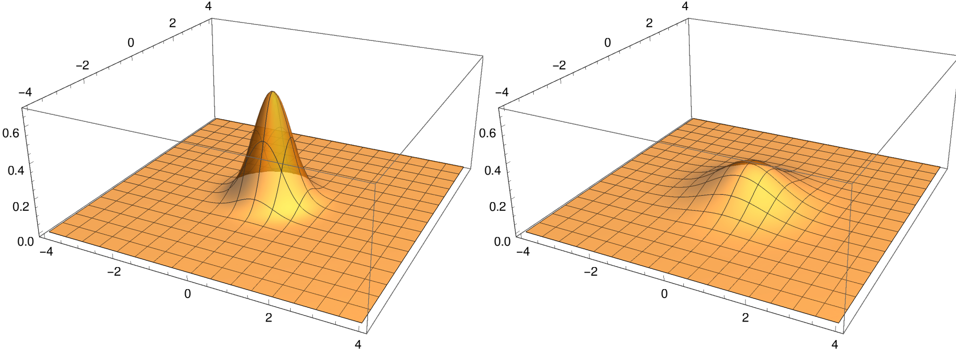
vacuum



$n = 5$

Wigner function

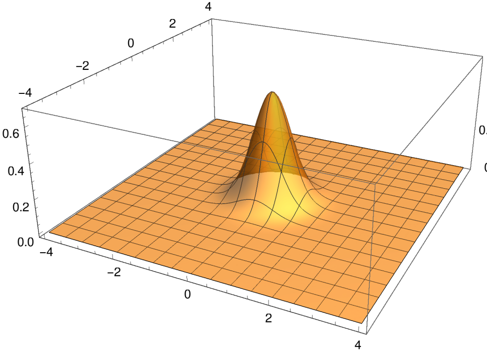
Thermal state



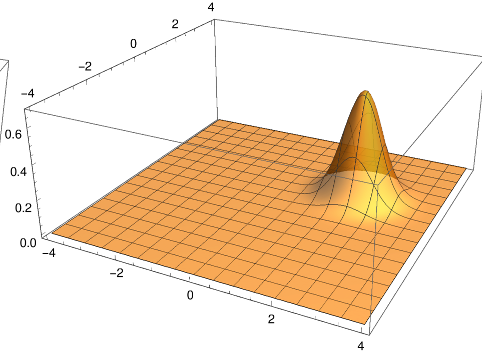
vacuum $\qquad$ $\bar{n} = 1$

Wigner function

Coherent state



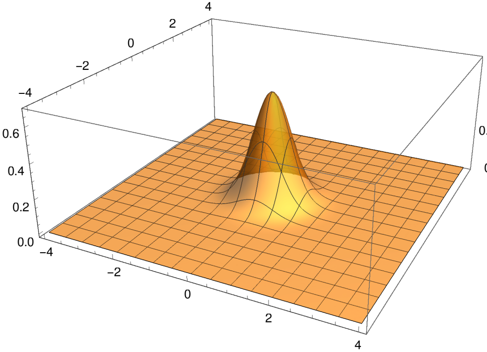
vacuum



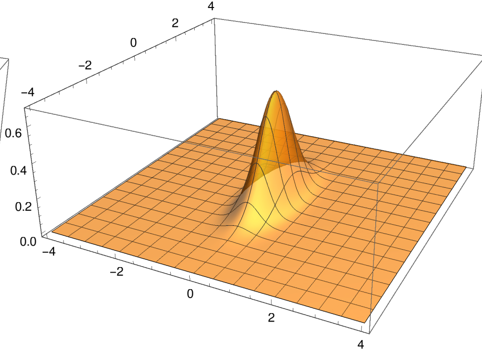
$\bar{n} = |\alpha|^2 = 5$

Wigner function

Squeezed state



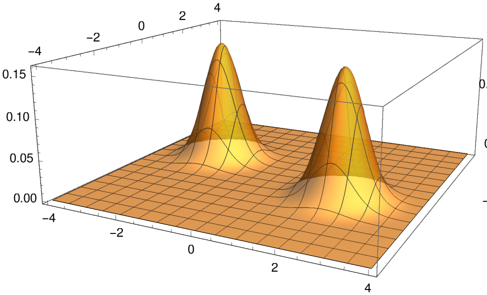
vacuum



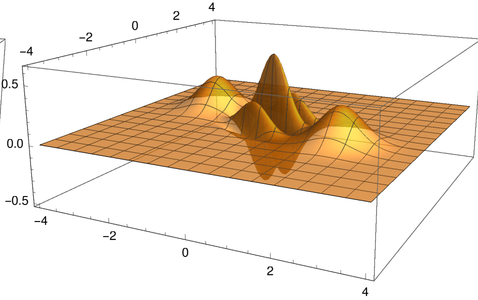
$\eta = 0.5$

Wigner function

Cat state vs mixture



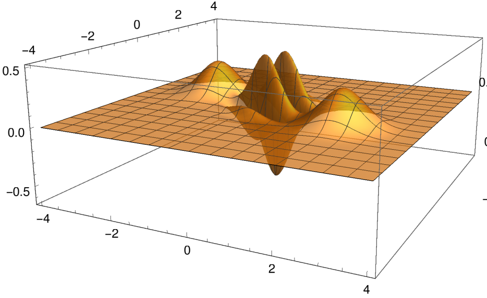
mixture $\rho = \frac{1}{2}\left(|\beta\rangle\langle\beta| + |-\beta\rangle\langle-\beta|\right)$



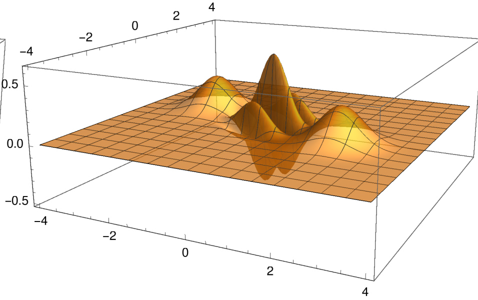
cat $\propto |\beta\rangle + |-\beta\rangle$

Wigner function

Cat states



$\text{cat} \propto |\beta\rangle - |-\beta\rangle$



$\text{cat} \propto |\beta\rangle + |-\beta\rangle$