Atoms and photons

Course 1

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adapted from Jean-Michel Raimond's slides

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Introduction

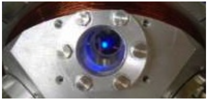
The fundamental importance of the atom-field interaction problem

- Provides all information we have on the universe except gravitational waves, which requires (quantum) optics
- Provides the most precise theory so far: QED (ex: theory/experiment comparisons for α or h/m, search for variation of constants (α, m_e/m_p…)
- Provides the best tests of fundamental quantum physics (ex: Bell inequalities, non-locality…)

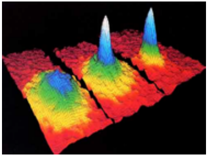
Introduction

The practical importance of the atom-field interaction problem

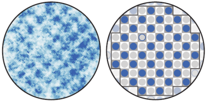
- Lasers
- Atomic clocks
- Cold atoms and BEC
- Quantum simulation
- Entanglement used as a resource (quantum spectroscopy, quantum information...)



Sr clock



BEC



AF state with cold fermions

Outline of this course

Chapter 1: Interaction of atoms with a classical field

1. The harmonically bound electron: a surprisingly successful model
2. The Einstein coefficients

Outline of this course

Chapter 2: Quantized atom and classical field

1. Interaction Hamiltonian
2. Free atom and resonant field
3. Relaxing atom and resonant field
4. Optical Bloch equations
5. Applications of the optical Bloch equations

Outline of this course

Chapter 3: Field quantization

1. Field eigenmodes
2. Quantization
3. Field quantum states
4. Field relaxation

Outline of this course

Chapter 4: quantized matter and quantized field

1. Interaction Hamiltonian
2. Spontaneous emission
3. Photodetection
4. The dressed atom
5. Applications of quantum optics (CQED = Cavity Quantum ElectroDynamics, squeezing for precision measurements, quantum simulation. . .)

Bibliography

- lecture notes by J.-M. Raimond (or C. Fabre in French) and slides handouts.
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Online lecture notes

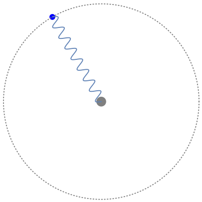
- www-lpl.univ-paris13.fr/bec, following the menu items Group members / Hélène Perrin
- C. Fabre M2 lecture notes (in French)

Outline

I. A classical model: the harmonically bound electron

1) The model

The simplest classical model for an atom: a single charge (electron) bound to a force center by an harmonic potential.



- An early atomic theory model (Thomson's 'plum pudding')
- A good guide to identify relevant parameters by dimensional analysis
- Surprisingly accurate predictions

A classical model: the harmonically bound electron

Equations of motion

Dynamics

$$\frac{d^2\mathrm{r}}{dt^2} + \omega_0^2 \mathrm{r} = 0 \qquad (1)$$

Solution

$$\mathrm{r} = \mathrm{r}_0 \exp(-i\omega_0 t) \qquad (2)$$

Natural units for the Bohr atom

Use the natural units $\hbar$, m (electron mass), c: energies in mc^2, frequencies in $\frac{mc^2}{\hbar}$, times in $\dfrac{\hbar}{mc^2} = 1.3\,10^{-21}$ s, lengths in $\dfrac{\hbar}{mc} = 3.86\,10^{-13}$ m

- Bohr radius: orbit with angular momentum $mva_0 = \hbar$

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{mq^2} = \frac{4\pi\epsilon_0\hbar c}{q^2}\frac{\hbar}{mc} \quad \text{i.e.} \quad \boxed{a_0 = \frac{1}{\alpha}\frac{\hbar}{mc}} = 5.3\times10^{-11}\text{ m}$$

- We introduced the fine structure constant

$$\boxed{\alpha = \frac{q^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}} \tag{3}$$

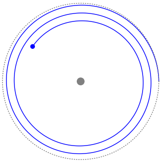
- The binding / ionization energy is the Rydberg constant $R_y = -E_{\rm tot} = E_{\rm kin} = mv^2/2$, and $\omega_0 \sim R_y/\hbar \sim 10^{16}$ s^{-1}

$$R_y = \frac{1}{2}\frac{q^2}{4\pi\epsilon_0 a_0} = \frac{1}{2}\frac{\hbar c\alpha}{a_0} = \frac{\alpha^2}{2}mc^2 \qquad \boxed{v = \alpha c \ll c, \quad \omega_0 = \frac{\alpha^2}{2}\frac{mc^2}{\hbar}}$$

A classical model: the harmonically bound electron

Damping

Damping: radiation reaction. Model: power emitted by a viscous damping term in the equation of motion. A reasonable approximation for weak damping.



Larmor formula for radiated power

for non relativistic motion: $$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} = m\tau a^2 \quad (4)$$

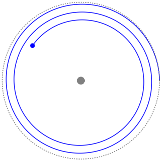
where $$\boxed{\tau = \frac{1}{6\pi\epsilon_0}\frac{q^2}{mc^3}} = 6.3 \times 10^{-24}\ \text{s} \quad (5)$$

related to the classical radius of electron $\boxed{r_e = \frac{q^2}{4\pi\epsilon_0 mc^2} = \alpha\frac{\hbar}{mc}}$

by $\boxed{\tau = \frac{2}{3}\frac{r_e}{c} = \frac{2\alpha}{3}\frac{\hbar}{mc^2}}$ N.B. $r_e = 2.8 \times 10^{-15}$ m

A classical model: the harmonically bound electron

Damping coefficient



friction force: $\mathbf{F} = -m\gamma\mathbf{v}$
dissipated power: $P = -\mathbf{F}\cdot\mathbf{v} = m\gamma v^2 = m\tau a^2 \sim m\tau\omega_0^2 v^2$
$\Rightarrow$ relevant damping rate $\gamma = \omega_0^2\tau$

Modified equation of motion

$$\frac{d^2\mathbf{r}}{dt^2} + \gamma\frac{d\mathbf{r}}{dt} + \omega_0^2\mathbf{r} = 0 \tag{6}$$

with

$$\boxed{\gamma = \omega_0^2\tau} \tag{7}$$

being the amplitude damping coefficient obtained by equalling the average dissipated power to the average radiated power (the energy damping coefficient is 2γ).

Order of magnitude for damping

- Typical transition frequency $\omega_0 = \frac{\alpha^2}{2}\frac{mc^2}{\hbar}$
- $\tau = \frac{2}{3}\frac{r_e}{c} = \frac{2\alpha}{3}\frac{\hbar}{mc^2}$

Order of magnitude estimate for $\gamma/\omega_0 = \omega_0\tau$:

$$\frac{\gamma}{\omega_0} = \omega_0\tau = \frac{\alpha^3}{3} \approx 1.3\,10^{-7} \tag{8}$$

$\Rightarrow$ weak damping, quasi constant orbits

I. A classical model: the harmonically bound electron

2) Polarizability

Response to a classical oscillating field $E_0 \mathbf{u}_z \exp(-i\omega t)$

Equation of motion

$$\frac{d^2\mathbf{r}}{dt^2} + \gamma \frac{d\mathbf{r}}{dt} + \omega_0^2 \mathbf{r} = \frac{qE_0}{m} \mathbf{u}_z e^{-i\omega t} \tag{9}$$

Steady-state solution

Position: $\mathbf{r} = \mathbf{r}_0 \exp(-i\omega t)$; Dipole: $\mathbf{d} = \mathbf{d}_0 \exp(-i\omega t)$ with

$$\mathbf{d}_0 = q\mathbf{r}_0 = \epsilon_0 \alpha_c E_0 \mathbf{u}_z \tag{10}$$

where

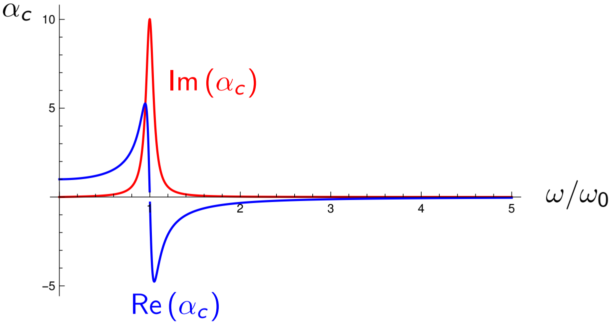
$$\alpha_c = \frac{q^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{11}$$

A classical model: the harmonically bound electron

Polarizability

N.B.: $\dfrac{q^2}{m\epsilon_0\omega_0^2} = 4\pi\alpha\dfrac{\hbar c}{m}\dfrac{1}{\omega_0^2} \sim \dfrac{16\pi}{\alpha^3}\left(\dfrac{\hbar}{mc}\right)^3 = 16\pi a_0^3$ Bohr volume

$$\alpha_c = \frac{q^2}{m\epsilon_0\omega_0^2}\,\frac{1}{1-\omega^2/\omega_0^2 - i\gamma\omega/\omega_0^2}$$



I. A classical model: the harmonically bound electron

3) Scattering regimes of incident power

Total power scattered by the atom given by Larmor formula:

$$\mathcal{P} = m\tau\overline{a^2} = \frac{1}{2}m\tau\omega^4|r_0|^2 \tag{12}$$

or, using $m\tau/q^2 = 1/(6\pi\epsilon_0 c^3)$

$$\mathcal{P} = \frac{|d_0|^2\omega^4}{12\pi\epsilon_0 c^3} = \frac{|\alpha_c|^2}{12\pi}\frac{\omega^4}{c^4}\epsilon_0 c E_0^2 \sim \alpha^2\epsilon_0 c E_0^2 \tag{13}$$

Cross Section

Ratio of this power to the incident power per unit surface $\mathcal{P}_i = \epsilon_0 c E_0^2/2$:

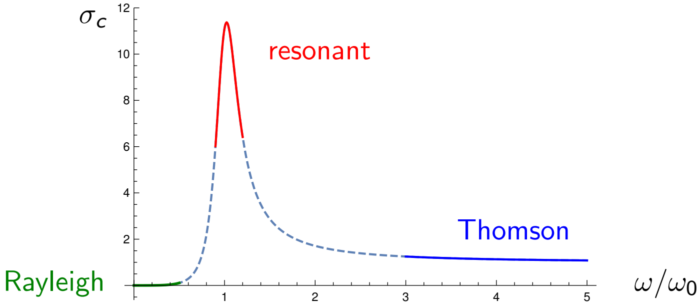
$$\sigma_c = \frac{1}{6\pi}\left(\frac{\omega}{c}\right)^4|\alpha_c|^2 = \frac{8\pi}{3}r_e^2\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \tag{14}$$

Light scattering cross section

Three regimes

Cross Section

$$\sigma_c = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$



Light scattering cross section

The three scattering regimes

Rayleigh scattering for $\omega < \omega_0$ and $\omega_0 - \omega \gg \gamma$

$$\sigma_c = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{\omega_0^4} \tag{15}$$

Blue sky: $\sigma_c \approx 10^{-30}$ m^2, density $\rho = 10^{25}$ m^{-3}: the attenuation length is $L = 1/\rho\sigma_c \approx 100$ km

Thomson scattering for $\omega > \omega_0$ and $\omega_0 - \omega \gg \gamma$

$$\sigma_c = \frac{8\pi}{3} r_e^2 \ . \tag{16}$$

Resonant regime for $\omega \approx \omega_0$

$$\boxed{\sigma_c = \frac{8\pi}{3} r_e^2 \frac{\omega_0^2}{4(\omega_0 - \omega)^2 + \gamma^2}} \tag{17}$$

A classical model: the harmonically bound electron

Resonant scattering

At exact resonance $\omega_0 = \omega$:

$$\sigma_c = \frac{8\pi}{3} r_e^2 \frac{\omega_0^2}{\gamma^2} \tag{18}$$

with

$$r_e \frac{\omega_0}{\gamma} = \frac{3}{2} c\tau \frac{1}{\omega_0 \tau} = \frac{3}{4\pi} \lambda_0 \tag{19}$$

where $\lambda_0 = 2\pi c/\omega_0$ is the wavelength. Hence

$$\boxed{\sigma_c = \frac{3}{2\pi} \lambda_0^2} \tag{20}$$

This model does not apply for high powers: saturation (about 1 mW/cm^2). A quantum effect. More on that in next Chapter.

I. A classical model: the harmonically bound electron

4) Propagation in matter

Apply the model to propagation in matter. Simplifiying hypothesis:

- Consider harmonic plane wave
- Linear response theory
- Dilute matter: no difference between local and global field

Electric displacement

$$\mathrm{D} = \epsilon_0 \mathrm{E} + \mathrm{P}$$

P: density of polarization.
Dilute matter (independent scatterers): linear response
$\mathrm{P} = \epsilon_0 \chi_c \mathrm{E}$ with $\chi_c = \rho \alpha_c \Rightarrow \mathrm{D} = \epsilon_0 \epsilon_r \mathrm{E}$
$\epsilon_r = 1 + \chi_c = 1 + \rho \alpha_c$.

Dispersion relation

Equation of propagation

$$\Delta \mathrm{E} + \frac{\omega^2}{c^2}\epsilon_r \mathrm{E} = 0 \qquad (21)$$

Dispersion relation

$$k^2 = k_0^2 \epsilon_r \qquad (22)$$

where $k_0 = \omega/c$

Refraction index

$$n = \sqrt{\epsilon_r} = n' + i n'' \qquad (23)$$

Refraction index

$$n' = \frac{1}{\sqrt{2}}\sqrt{\epsilon_r' + \sqrt{\epsilon_r'^2 + \epsilon_r''^2}} \quad \text{and} \quad n'' = \frac{\epsilon_r''}{\sqrt{2}}\frac{1}{\sqrt{\epsilon_r' + \sqrt{\epsilon_r'^2 + \epsilon_r''^2}}} \tag{24}$$

Real part: refraction (ordinary index).
Imaginary part: absorption. Density of power released in matter $\mathcal{P} = \frac{1}{2}\mathrm{Re}\, \mathrm{j}_0 \cdot \mathrm{E}_0^*$ where $\mathrm{j}_0 = -i\omega \mathrm{P}_0$.

$$\mathcal{P} = \frac{1}{2}\mathrm{Re}\,(-i\omega \mathrm{P}_0 \cdot \mathrm{E}_0^*) = \frac{1}{2}\mathrm{Re}\,(-i\chi_c)\,\epsilon_0\omega|\mathrm{E}_0|^2 \tag{25}$$

$$\mathcal{P} = \frac{1}{2}\epsilon_0\omega\chi''|E_0|^2 = \frac{1}{2}\epsilon_0\omega\rho\alpha_c''|E_0|^2 \tag{26}$$

A classical model: the harmonically bound electron

Propagation in matter

$$\mathcal{P} = \frac{1}{2}\epsilon_0\omega\chi''|E_0|^2 = \frac{1}{2}\epsilon_0\omega\rho\alpha_c''|E_0|^2 \tag{27}$$

Imaginary part of polarizability:

$$\boxed{\alpha_c'' = \frac{q^2}{m\epsilon_0}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} > 0 \tag{28}$$

Classical appraoch predicts that power released is always positive, matter is always absorbing. Laser needs a quantum ingredient.

II. Einstein's coefficents

1) Introduction - The coefficients

A phenomenological description of energy exchanges between light and matter. A very simple description:

- Field only described by its spectral energy density u_ν.
  Number density of photons between ν and $\nu + d\nu$: $u_\nu / h\nu$.
  Total energy per unit volume: $u = \int u_\nu \, d\nu$
- Matter made of non degenerate two-level atoms, $g \to e$, energies E_g and E_e with $(E_e - E_g) = h\nu_0$. $\lambda_0 = c/\nu_0$.
- Number (or density) of atoms in the two levels Π_e and Π_g, normalized to the total atom number (or density ρ) so that $\Pi_e + \Pi_g = 1$.

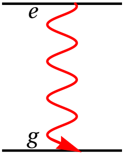
Goal: obtain rate equations for the variations of Π_e and u_ν. We consider in particular the radiation/matter thermal equilibrium at a temperature T. For that, three processes come into play:

Three processes

Spontaneous emission

Deexcitation of e with a constant probability per unit time, $A_{eg} = \Gamma$.

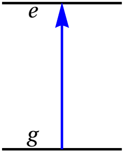
$$\left.\frac{d\Pi_e}{dt}\right)_{\text{spont}} = -A_{eg}\Pi_e \tag{29}$$



Absorption

Transfer from g to e by absorption of photons. Rate proportional to the photon density (a cross-section approach).

$$\left.\frac{d\Pi_e}{dt}\right)_{\text{abs}} = B_{ge}u_{\nu_0}\Pi_g \tag{30}$$



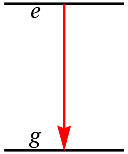
Three processes

Absorption and spontaneous emission are not enough: at infinite temperature, $u_\nu \to \infty$, $\Pi_e \to 1$. Not the prediction of thermodynamics ($\Pi_e = \Pi_g = 0.5$). Einstein adds a third process:

Stimulated emission

Transition from e to g and emission of a photon at a rate proportional to the photon density.

$$\left.\frac{d\Pi_e}{dt}\right)_{\text{stim}} = -B_{eg}\, u_{\nu_0} \Pi_e \qquad (31)$$



Einstein's rate equations

$$\boxed{\frac{d\Pi_e}{dt} = -A_{eg}\Pi_e - B_{eg}\, u_{\nu_0}\Pi_e + B_{ge}\, u_{\nu_0}\Pi_g} \qquad (32)$$

II. Einstein's coefficents

2) Relations between the three coefficients

At thermal equilibrium (temperature T)

$$\frac{\Pi_e}{\Pi_g} = e^{(E_g - E_e)/k_B T} = e^{-h\nu_0/k_B T} \tag{33}$$

k_B: Boltzmann constant. And (Planck's law)

$$\boxed{u_{\nu_0} = \frac{8\pi h{\nu_0}^3}{c^3}\frac{1}{\exp(h\nu_0/k_B T) - 1}} \tag{34}$$

Relations between the three coefficients

In steady state: $(A_{eg} + B_{eg} u_{\nu_0})\Pi_e = B_{ge} u_{\nu_0} \Pi_g$. For $T \to \infty$, $u_{\nu_0} \to \infty$ and $\Pi_e/\Pi_g \to 1$. Neglect spontaneous emission.

$$\boxed{B_{ge} = B_{eg} = B} \tag{35}$$

Noting $A_{eg} = A$, steady state at a finite temperature T:

$$A + B u_{\nu_0} = B u_{\nu_0} \frac{\Pi_g}{\Pi_e} = B u_{\nu_0} e^{h\nu_0/k_B T} \tag{36}$$

Hence

$$u_{\nu_0} = \frac{A}{B} \frac{1}{\exp(h\nu_0/k_B T) - 1} \tag{37}$$

Comparing with Planck's law

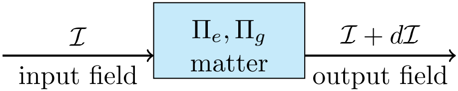
$$\boxed{\frac{A}{B} = \frac{8\pi h {\nu_0}^3}{c^3} = \frac{8\pi h}{\lambda_0^3}} \tag{38}$$

$\Rightarrow$ only need $A = \Gamma$ to get all three!

II. Einstein's coefficents

3) A consequence of stimulated emission: the laser

Stimulated emission: addition of energy to the incoming wave.
A simple situation: plane wave at frequency ν_0 on a thin slice of atoms. Incoming power per unit surface $\mathcal{I}$, outgoing $\mathcal{I} + d\mathcal{I}$.



balance: $d\mathcal{I} \propto \mathcal{I}(\Pi_e - \Pi_g) = \mathcal{I}\,\Delta$ where Δ is the population inversion density:

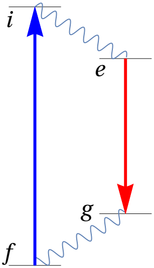
$$\Delta = \Pi_e - \Pi_g \tag{39}$$

The power increases when $\Delta > 0$: gain requires population inversion.

Population inversion

Conditions to achieve $\Delta > 0$

- No thermal equilibrium
- No two-level system (in the steady state)
- Three or four level system
- Case of a four level system (f: ground state, i intermediate, plus e and g:



- Fast incoherent pumping from f to i
- Fast relaxation from i to e
- Stimulated emission from e to g
- Extremely fast relaxation from g to f

The Laser

Principle

- Gain + feedback = oscillation
- A laser is composed of an amplifying medium (gain) and of an optical resonant cavity (feedback).
- When the gain exceeds the losses in the feedback, a self-sustained steady-state oscillation occurs.

The Laser: A simple model

Captures the main physical ideas without any complication. Forget about all details and proportionality constants.

Variables

- Population inversion density Δ. If g strongly damped, $\Delta = \Pi_e$.
- Intra-cavity intensity $\mathcal{I}$ (photon density)

A simple model

Evolution of intensity

$$\frac{d\mathcal{I}}{dt} = -\kappa\mathcal{I} + G\mathcal{I}\Delta \tag{40}$$

κ: rate of internal or coupling cavity losses.

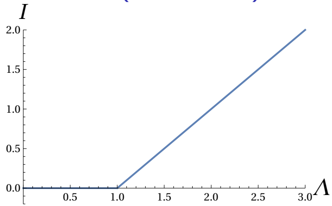
Evolution of population inversion

$$\frac{d\Delta}{dt} = \Lambda - \Gamma\Delta - G\mathcal{I}\Delta \tag{41}$$

with

- Λ : pumping rate in the upper level e
- Γ : relaxation rate of e (spontaneous emission in modes other than the cavity one, other sources of atomic losses)

Steady state: $\mathcal{I}(G\Delta - \kappa) = 0 \qquad \Lambda = \Delta(\Gamma + G\mathcal{I})$



Laser off solution

- $\mathcal{I} = 0$ always a solution
- $\Delta = \Lambda/\Gamma$

Laser on solution

- $\Delta = \kappa/G$. Possible only if $\Delta < 1$ i.e. κ (loss) $< G$ (gain)

$$\mathcal{I} = \frac{1}{\kappa}\left(\Lambda - \frac{\Gamma\kappa}{G}\right) \tag{42}$$

- Relevant if $\mathcal{I} \geq 0 \Rightarrow$ threshold condition

$$\boxed{\Lambda \geq \Lambda_t = \frac{\Gamma\kappa}{G}} \tag{43}$$

The Laser: Stability of the solutions

- $\Lambda < \Lambda_t$: only solution $\mathcal{I} = 0$
- $\Lambda \geq \Lambda_t$: two possible solutions, but $\mathcal{I} = 0$ unstable