

Solution of the problem

Sisyphus effect at low saturation on a $J_g = 1/2 \leftrightarrow J_e = 3/2$ transition

1 Model system

1.1 Laser field configuration

1. The total field can be written as:

$$\begin{aligned}
 \mathbf{E}_L(z, t) &= \frac{1}{2} \mathcal{E}_0 \left(\mathbf{e}_x e^{i(kz - \omega_L t)} - i \mathbf{e}_y e^{i(-kz - \omega_L t)} + c.c. \right) \\
 &= \frac{1}{2} \mathcal{E}_0 \left(e^{-i\omega_L t} (\mathbf{e}_x e^{ikz} - i \mathbf{e}_y e^{-ikz}) \right) + c.c. \\
 &= \frac{1}{2} \mathcal{E}_0 \left(e^{-i\omega_L t} ((\mathbf{e}_x - i \mathbf{e}_y) \cos kz + i(\mathbf{e}_x + i \mathbf{e}_y) \sin kz) \right) + c.c. \\
 &= \frac{1}{2} \mathcal{E}_0 \sqrt{2} \boldsymbol{\epsilon}(z) e^{-i\omega_L t} + c.c. \tag{1}
 \end{aligned}$$

The field amplitude is $\mathcal{E}_L = \sqrt{2} \mathcal{E}_0$, and the polarisation is $\boldsymbol{\epsilon}(z) = \cos kz \boldsymbol{\epsilon}_- - i \sin kz \boldsymbol{\epsilon}_+$ as expected.

2. The polarisation is σ_- in $z = 0$, linear along $(\mathbf{e}_x - \mathbf{e}_y)/\sqrt{2}$ in $z = \lambda/8$, σ_+ in $z = \lambda/4$, linear along $(\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$ in $z = 3\lambda/8$, again σ_- in $z = \lambda/2$, etc. There is a strong polarisation gradient, *i.e.* the polarisation varies on a short scale with a period $\lambda/2$.

1.2 Dipole force

1. We calculate the action of the operator $(\boldsymbol{\epsilon}(z) \cdot \hat{\mathbf{d}}^+)$ on the two ground states $|g, +1/2\rangle$ and $|g, -1/2\rangle$:

$$(\boldsymbol{\epsilon}(z) \cdot \hat{\mathbf{d}}^+) |g, +1/2\rangle = \frac{1}{\sqrt{3}} \cos(kz) |e, -1/2\rangle - i \sin(kz) |e, +3/2\rangle, \tag{2}$$

$$(\boldsymbol{\epsilon}(z) \cdot \hat{\mathbf{d}}^+) |g, -1/2\rangle = \cos(kz) |e, -3/2\rangle - i \frac{1}{\sqrt{3}} \sin(kz) |e, +1/2\rangle. \tag{3}$$

2. The two states calculated above are orthogonal. As Λ is obtained from the operator $\boldsymbol{\epsilon}(z) \cdot \hat{\mathbf{d}}^+$ and its hermitian conjugate, we deduce the following matrix elements of Λ in the basis $(|g, -1/2\rangle, |g, +1/2\rangle)$:

$$\Lambda_{++} = \langle g, +1/2 | \Lambda | g, +1/2 \rangle = \frac{1}{3} \cos^2 kz + \sin^2 kz = 1 - \frac{2}{3} \cos^2 kz, \tag{4}$$

$$\Lambda_{--} = \langle g, -1/2 | \Lambda | g, -1/2 \rangle = \cos^2 kz + \frac{1}{3} \sin^2 kz = 1 - \frac{2}{3} \sin^2 kz, \quad (5)$$

$$\Lambda_{+-} = \langle g, +1/2 | \Lambda | g, -1/2 \rangle = 0 = \Lambda_{-+}. \quad (6)$$

The states $|g, \pm 1/2\rangle$ are thus eigenstates of Λ with the eigenvalues given above.

3. The dipole force is obtained by differentiating the eigenenergies:

$$\mathcal{F}_{\text{react}} = -\Pi_{+1/2} \nabla E_{+1/2} - \Pi_{-1/2} \nabla E_{-1/2}. \quad (7)$$

1.2.1 Dissipative force

For an atom at rest, with a low saturation parameter and with orthogonal polarisations, the dissipative force is simply the sum of the radiation pressure of both beams. With two beams with equal intensities, this total force is zero for both internal states.

2 Dynamics of the internal degrees of freedom

2.1 Light shifts in the ground state

1. The light shifts are the eigenvalues of the operator $H_{\text{eff}} = \hbar \delta' \Lambda$, which is diagonal in the basis $|g, \pm 1/2\rangle$. Here $\hbar \delta' = \hbar \delta s / 2 = \hbar \delta s_0$, where s_0 is the saturation parameter for a single beam. The light shifts are thus:

$$E_{\pm 1/2}(z) = \hbar \delta s_0 \Lambda_{\pm\pm}(z) = -\frac{3}{2} U_0 \Lambda_{\pm\pm}(z). \quad (8)$$

From the eigenvalues calculated previously, we obtain:

$$E_{+1/2}(z) = U_0 \left(-\frac{3}{2} + \cos^2 kz \right) \quad \text{and} \quad E_{-1/2}(z) = U_0 \left(-\frac{3}{2} + \sin^2 kz \right). \quad (9)$$

2. The mean force is then:

$$\begin{aligned} \mathcal{F} &= U_0 \left(\Pi_{+1/2}(z) 2k \cos kz \sin kz - \Pi_{-1/2}(z) 2k \cos kz \sin kz \right) \mathbf{e}_z \\ &= k U_0 \mathcal{M}(z) \sin 2kz \mathbf{e}_z. \end{aligned} \quad (10)$$

2.2 Optical pumping rate

1. The calculation of the optical pumping rates is the difficult point of this problem... The time evolution of the population $\Pi_{+1/2}(z)$ is governed by a rate equation. The population of the state $|g, +1/2\rangle$ varies due to a gain from the state $|g, -1/2\rangle$, proportional to $\Pi_{-1/2}(z)$, and losses towards $|g, -1/2\rangle$, proportional to $\Pi_{+1/2}(z)$. The coefficients in front of the populations are the optical pumping rates.

The departure rates are directly linked to the eigenvalues of Λ :

$$\Gamma'_{+1/2}(z) = \Gamma' \langle g, +1/2 | \Lambda | g, +1/2 \rangle = \Gamma' \left(1 - \frac{2}{3} \cos^2 kz \right) \quad (11)$$

$$\Gamma'_{-1/2}(z) = \Gamma' \langle g, -1/2 | \Lambda | g, -1/2 \rangle = \Gamma' \left(1 - \frac{2}{3} \sin^2 kz \right). \quad (12)$$

The arrival rate are more painful to obtain. Let us calculate the arrival rate on state $|g, m\rangle$.

$$\Gamma_m''(z) = \Gamma' \langle g, +1/2 | \sum_{q=-1,0,+1} (\epsilon_q^* \cdot \hat{d}^-) (\epsilon(z) \cdot \hat{d}^+) \sigma_{gg} (\epsilon^*(z) \cdot \hat{d}^-) (\epsilon_q \cdot \hat{d}^+) |g, +1/2\rangle. \quad (13)$$

We recall $\epsilon(z) = \cos kz \epsilon_- - i \sin kz \epsilon_+ = \cos kz \epsilon_{-1} - i \sin kz \epsilon_1$.

The additional information gives the action of $(\epsilon_q \cdot \hat{d}^+)$ on $|g, m\rangle$ as a function of the Clebsh-Gordan coefficient:

$$(\epsilon_q \cdot \hat{d}^+) |g, m\rangle = \langle J_e \ m + q | J_g \ 1 \ m \ q \rangle |e, m + q\rangle, \quad \text{and,}$$

$$\langle g, m | (\epsilon_q^* \cdot \hat{d}^-) = \langle J_e \ m + q | J_g \ 1 \ m \ q \rangle \langle e, m + q |.$$

Now, as $\epsilon^*(z) = \cos kz \epsilon_{-1}^* + i \sin kz \epsilon_1^*$, we have

$$\begin{aligned} & (\epsilon^*(z) \cdot \hat{d}^-) |e, m + q\rangle = \\ & = \cos kz (\epsilon_{-1}^* \cdot \hat{d}^-) |e, m + q\rangle + i \sin kz (\epsilon_1^* \cdot \hat{d}^-) |e, m + q\rangle \end{aligned} \quad (14)$$

$$\begin{aligned} & = \cos kz \langle J_e \ m + q | J_g \ 1 \ m + q + 1, -1 \rangle |g, m + q + 1\rangle \\ & + i \sin kz \langle J_e \ m + q | J_g \ 1 \ m + q - 1, 1 \rangle |g, m + q - 1\rangle. \end{aligned} \quad (15)$$

On the other side:

$$\begin{aligned} & \langle e, m + q | (\epsilon(z) \cdot \hat{d}^+) = \\ & = \cos kz \langle e, m + q | (\epsilon_{-1} \cdot \hat{d}^+) - i \sin kz \langle e, m + q | (\epsilon_1 \cdot \hat{d}^+) \end{aligned} \quad (16)$$

$$\begin{aligned} & = \cos kz \langle J_e \ m + q | J_g \ 1 \ m + q + 1, -1 \rangle \langle g, m + q + 1 | \\ & - i \sin kz \langle J_e \ m + q | J_g \ 1 \ m + q - 1, 1 \rangle \langle g, m + q - 1 |. \end{aligned} \quad (17)$$

The σ_{gg} operator gives the populations Π_{\pm} when taken between the same $|g, m\rangle$ states, and 0 otherwise. The cross terms then disappear, and the final result is:

$$\begin{aligned} \Gamma_m''(z) = \Gamma' \sum_{q=-1,0,+1} & \langle J_e \ m + q | J_g \ 1 \ m \ q \rangle^2 \times \\ & \left(\cos^2 kz \langle J_e \ m + q | J_g \ 1 \ m + q + 1, -1 \rangle^2 \Pi_{m+q+1} \right. \\ & \left. + \sin^2 kz \langle J_e \ m + q | J_g \ 1 \ m + q - 1, 1 \rangle^2 \Pi_{m+q-1} \right). \end{aligned} \quad (18)$$

Some of these terms are zero, if $|m + q \pm 1| > 1/2$. For $m = +1/2$, we obtain for the three terms:

$$\begin{aligned} q = 1: & \Gamma' \times 1 \times (0 + \sin^2 kz \times 1 \times \Pi_+) = \Gamma' \Pi_+ \sin^2 kz. \\ q = 0: & \Gamma' \times \frac{2}{3} \times (0 + \sin^2 kz \times \frac{1}{3} \times \Pi_-) = \frac{2}{9} \Gamma' \Pi_- \sin^2 kz. \\ q = -1: & \Gamma' \times \frac{1}{3} \times (\cos^2 kz \times \frac{1}{3} \times \Pi_+ + 0) = \frac{1}{9} \Gamma' \Pi_+ \cos^2 kz. \end{aligned}$$

It corresponds to the three ways of having an arrival to $|g, 1/2\rangle$: starting from $|g, 1/2\rangle$ and going back via the excited state $|e, 3/2\rangle$: Clebsh 1×1 ; starting from $|g, -1/2\rangle$ and arriving to $|g, 1/2\rangle$ through $|e, 1/2\rangle$: Clebsh $\frac{1}{3} \times \frac{2}{3}$; starting from $|g, 1/2\rangle$ and going back via the excited state $|e, -1/2\rangle$: Clebsh $\frac{1}{3} \times \frac{1}{3}$.

The total arrival rate to $|g, 1/2\rangle$ is finally

$$\Gamma''_{+1/2}(z) = \Gamma' \left(\left(1 - \frac{8}{9} \cos^2 kz\right) \Pi_+(z) + \frac{2}{9} \sin^2 kz \Pi_-(z) \right). \quad (19)$$

The same reasoning gives the arrival rate to the state $|g, -1/2\rangle$:

$$\Gamma''_{-1/2}(z) = \Gamma' \left(\left(1 - \frac{8}{9} \sin^2 kz\right) \Pi_-(z) + \frac{2}{9} \cos^2 kz \Pi_+(z) \right). \quad (20)$$

2. For the population $\Pi_+(z)$ we get the following differential equations:

$$\begin{aligned} \frac{d\Pi_+}{dt} &= \Gamma' \left(\left(1 - \frac{8}{9} \cos^2 kz\right) \Pi_+(z) + \frac{2}{9} \sin^2 kz \Pi_-(z) \right) - \Gamma' \left(1 - \frac{2}{3} \cos^2 kz\right) \Pi_+(z) \\ \frac{d\Pi_+}{dt} &= \frac{2}{9} \Gamma' \left(\sin^2 kz \Pi_-(z) - \cos^2 kz \Pi_+(z) \right). \end{aligned} \quad (21)$$

We define $\gamma = \frac{2}{9} \Gamma' = \frac{2}{9} s_0 \Gamma$ and recall that $\Pi_+(z) + \Pi_-(z) = 1$, and $\mathcal{M}(z) = \Pi_+(z) - \Pi_-(z) = 2\Pi_+(z) - 1$. We get:

$$\frac{d\Pi_+}{dt} = -\gamma \left(\cos^2 kz \Pi_+(z) - \sin^2 kz \Pi_-(z) \right) = -\gamma \left(\Pi_+(z) - \sin^2 kz \right) \quad (22)$$

$$\frac{d\mathcal{M}}{dt} = -\gamma \left(\mathcal{M}(z) + 1 - 2 \sin^2 kz \right) = -\gamma \left(\mathcal{M}(z) + \cos(2kz) \right). \quad (23)$$

We recover the proposed equation, with $\tau_P = \gamma^{-1}$. This time is the typical pumping time between states $|g, 1/2\rangle$ and $|g, -1/2\rangle$, *i.e.* the time to reach the steady state.

3. In the steady state, $\frac{d\mathcal{M}}{dt} = 0$ and $\mathcal{M}(z) = -\cos(2kz) = 2 \sin^2 kz - 1$. The population in the two ground states are then $\Pi_+(z) = \sin^2 kz$ and $\Pi_-(z) = \cos^2 kz$. There is a correlation between populations and light shifts: the population is always largest in the state which has a lower potential energy.

4. For a kinetic energy much larger than the well depth ($Mv^2/2 \gg U_0$), the atoms are not trapped. With a velocity $kv \gg \Gamma'$, it covers several wells before being pumped. The condition $kv \ll \Gamma$ means that the atom doesn't move during the pumping process, which can be considered as instantaneous. A pumping process occurs more likely when the atom is at the top of a hill, which removes an energy of order U_0 with a rate γ . This gives an idea of the cooling power in Sisyphus cooling.

The order of magnitude of the final temperature is the well depth: $k_B T \simeq U_0 \simeq \hbar \Omega_1^2 / |\delta|$ for large negative detunings.

3 Cooling mechanism for a moving atom

3.1 Characteristic times

1. The time scale of the evolution of the internal variables is $\tau_P = 9/(2\gamma s_0)$.
2. At the bottom of on the the wells, say, for the state $|g, -1/2\rangle$ around $z = 0$, the energy can be approximated by $E_-(z) \simeq -3U_0/2 = k^2U_0z^2$. This corresponds to a harmonic oscillator $M\Omega_{\text{osc}}^2z^2/2$ with a frequency

$$\Omega_{\text{osc}} = k\sqrt{\frac{2U_0}{M}} = \sqrt{\frac{4\hbar|\delta|s_0}{3M}}. \quad (24)$$

The typical external time is then $t_{\text{ext}} = \Omega_{\text{osc}}^{-1}$.

N.B. The oscillation frequency can be written equivalently as $\hbar\Omega_{\text{osc}} = 2\sqrt{U_0E_{\text{rec}}}$.

3. The assumption of hoping regime allows to consider that the atoms move slowly as compared to the pumping time. In this case, we can eliminate the internal variables adiabatically from the equation of motion. This semi-classical treatment allows the introduction of a force and a diffusion in momentum space, as in the case of Doppler cooling.

3.2 The hoping regime

1. With the assumption $t_{\text{int}} \ll t_{\text{ext}}$ we can write $z = vt$ (constant velocity) on the time scale τ_P .
2. The differential equation for $\mathcal{M}(t)$ is now:

$$\frac{d}{dt}\mathcal{M}(t) + \frac{1}{\tau_P}\mathcal{M}(t) = -\frac{1}{\tau_P}\cos 2kvt. \quad (25)$$

The forced solution is:

$$\mathcal{M}(t) = -\text{Re} \frac{1}{1 + 2ik\tau_P v} e^{2ikvt} \quad (26)$$

or:

$$\mathcal{M}(t) = -\frac{1}{1 + (v/v_c)^2} \cos 2kz - \frac{v/v_c}{1 + (v/v_c)^2} \sin 2kz. \quad (27)$$

3. The force averaged over one period $\overline{\mathcal{F}_z(v)} = \overline{kU_0\mathcal{M}(t)\sin(2kz)}$ is given by:

$$\overline{\mathcal{F}_z(v)} = -\frac{kU_0}{2} \frac{v/v_c}{1 + (v/v_c)^2} = -\alpha_S \frac{v}{1 + (v/v_c)^2} \quad (28)$$

with

$$\alpha_S = k^2U_0\tau_P = -3\hbar k^2 \frac{\delta}{\Gamma}. \quad (29)$$

For velocities $v \ll v_c$, we obtain a friction force with a friction coefficient α_S .

4. The friction coefficient α_S doesn't depend on the laser intensity (the trap depth decreases when s_0 decreases, but the pumping rate is lowered accordingly and the two effects compensate exactly). At large detuning and low intensity, α_S is always larger than the Doppler friction coefficient α_D , Sisyphus cooling being then more efficient. v_c is the capture velocity, at which the force is largest.

3.3 Equilibrium temperature

1. At low saturation, we have seen in lecture 2 that $D_R \simeq \frac{\Gamma}{2} s_0 \hbar^2 k^2$ for $s = 2s_0$.
2. The spatial average $\overline{D_{\text{dip}}}$ of the diffusion coefficient is

$$D_{\text{dip}} = \frac{3}{4} \hbar^2 k^2 \frac{\delta^2}{\Gamma} s_0. \quad (30)$$

3. At large detunings, $\overline{D_{\text{dip}}} \gg D_R$. The limit temperature of 1D Sisyphus cooling is then:

$$k_B T = \frac{D}{\alpha_S} \simeq -\frac{1}{4} \hbar \delta s_0 = \frac{3}{8} U_0 \simeq \frac{\hbar \Omega_1^2}{8|\delta|}. \quad (31)$$