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Hélène Perrin and Pierre Desbiolles

The multi-level atom

The dipolar interaction between atom and light field, in the rotating wave approximation, is given by

$$V_{AL} = -\mathbf{d}^+ \cdot \mathbf{E}_L^+(\mathbf{r}) e^{-i\omega_L t} - \mathbf{d}^- \cdot \mathbf{E}_L^-(\mathbf{r}) e^{+i\omega_L t} \tag{1}$$

with

$$\mathbf{d}^+ = P_e \mathbf{d} P_g \quad \text{and} \quad \mathbf{d}^- = P_g \mathbf{d} P_e \tag{2}$$

and P_g and P_e are the projection operators on the subspaces of the ground state and the excited state, respectively :

$$P_g = \sum_{m=-J_g}^{+J_g} |J_g m\rangle\langle J_g m| \quad \text{and} \quad P_e = \sum_{\mu=-J_e}^{+J_e} |J_e \mu\rangle\langle J_e \mu| \tag{3}$$

The fields $\mathbf{E}_L^+$ and $\mathbf{E}_L^-$ are defined by :

$$\mathbf{E}_L = \mathbf{E}_L^+ e^{-i\omega_L t} + \mathbf{E}_L^- e^{+i\omega_L t} \tag{4}$$

The polarisation vector $\boldsymbol{\epsilon}(\mathbf{r})$ at position $\mathbf{r}$ is linked to the field through :

$$\mathbf{E}_L^+(\mathbf{r}) = \frac{1}{2}\boldsymbol{\epsilon}(\mathbf{r})\mathcal{E}(\mathbf{r}) \quad \text{and} \quad \mathbf{E}_L^-(\mathbf{r}) = \frac{1}{2}\boldsymbol{\epsilon}^\star(\mathbf{r})\mathcal{E}(\mathbf{r}) \tag{5}$$

where we chose $\mathcal{E}(\mathbf{r})$ real. The complex polarisation is normalised as follows :

$$\boldsymbol{\epsilon}^\star(\mathbf{r}) \cdot \boldsymbol{\epsilon}(\mathbf{r}) = 1 \tag{6}$$

The polarisation can be decomposed on the normal basis :

$$\boldsymbol{\epsilon}_{\pm 1} = \boldsymbol{\epsilon}_\pm = \mp\frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y) \quad \text{and} \quad \boldsymbol{\epsilon}_0 = \mathbf{e}_z \tag{7}$$

corresponding respectively to the polarisations $\sigma^\pm$ and π. In order to evaluate the matrix elements of the operator V_{AL}, we should first calculate the matrix elements of the operators $\boldsymbol{\epsilon}_q \cdot \mathbf{d}^+$ and $\boldsymbol{\epsilon}_q^\star \cdot \mathbf{d}^-$. Apply the Wigner-Eckart theorem to the vectorial operator $\mathbf{d}$ gives :

$$\langle J_e \mu | \boldsymbol{\epsilon}_q \cdot \mathbf{d}^+ | J_g m\rangle = \mathcal{D}\langle J_e\ \mu | J_g\ 1\ m\ q\rangle \quad \text{and} \quad \langle J_g m | \boldsymbol{\epsilon}_q^\star \cdot \mathbf{d}^- | J_e \mu\rangle = \mathcal{D}\langle J_e\ \mu | J_g\ 1\ m\ q\rangle \tag{8}$$

where $\langle J_e \mu | J_g 1 m q\rangle$ is a Clebsh-Gordan coefficient and $\mathcal{D}$ is the reduced matrix element, which can always be chosen real and is independent of the magnetic quantum numbers μ, m and q. Recall that $\langle J_e \mu | J_g 1 m q\rangle$ is non zero if and only if $\mu = m + q$. We then define the operators $\hat{\mathbf{d}}^+$ and $\hat{\mathbf{d}}^- = (\hat{\mathbf{d}}^+)^\dagger$ through :

$$\mathbf{d}^+ = \mathcal{D}\hat{\mathbf{d}}^+ \quad \text{and} \quad \mathbf{d}^- = \mathcal{D}\hat{\mathbf{d}}^- \tag{9}$$

such that the matrix elements of $\boldsymbol{\epsilon}_q \cdot \hat{\mathbf{d}}^+$ and $\boldsymbol{\epsilon}_q^\star \cdot \hat{\mathbf{d}}^-$ are simply the Clebsh-Gordan coefficients associates to the corresponding transitions. We define the Rabi frequency at point $\mathbf{r}$ by :

$$\hbar\Omega_1(\mathbf{r}) = -\mathcal{D}\mathcal{E}(\mathbf{r}) \tag{10}$$

which corresponds to a transition with a Clebsh-Gordan coefficient equal to 1, excited by a laser field of amplitude $\mathcal{E}(\mathbf{r})$. Finally, the atom-laser coupling reads :

$$V_{AL} = \frac{\hbar\Omega_1}{2}\left(\hat{\mathbf{d}}^+ \cdot \boldsymbol{\epsilon}(\mathbf{r}) e^{-i\omega_L t} + \hat{\mathbf{d}}^- \cdot \boldsymbol{\epsilon}^\star(\mathbf{r}) e^{+i\omega_L t}\right) \tag{11}$$

similar, but not identical, to the two-level case.

In practical cases, the laser polarisation $\boldsymbol{\epsilon}$ has to be decomposed in the standard basis $(\boldsymbol{\epsilon}_\pm, \boldsymbol{\epsilon}_0)$ in order to calculate easily the matrix elements of V_{AL} from the Clebsh-Gordan coefficients of the transition.